THE SEPTEMBER MEETING OF THE SAN FRANCISCO SECTION.

THE twelfth regular meeting of the San Francisco Section of the AMERICAN MATHEMATICAL SOCIETY was held at the University of California on Saturday, September 28, 1907. The following members of the Society were present:

Professor R. E. Allardice, Professor H. F. Blichfeldt, Mr. A. J. Champreux, Professor R. L. Green, Professor M. W. Haskell, Professor L. M. Hoskins, Professor D. N. Lehmer, Mr. Joseph Lipke, Dr. J. H. McDonald, Professor W. A. Manning, Dr. T. M. Putnam, Professor Irving Stringham, Mr. J. D. Suter.

The following officers were elected for the ensuing year: Professor L. M. Hoskins, chairman; Professor W. A. Manning, secretary; Professor D. N. Lehmer, Dr. J. H. McDonald, Professor W. A. Manning, program committee. The next two meetings of the Section are to be held at Stanford University, February 29, 1908, and at the University of California, September 26, 1908.

The following papers were read at this meeting:

(1) Dr. J. H. MCDONALD: "On Minkowski's diagonal continued fraction."

(2) Professor H. F. BLICHFELDT: "Concerning the geodesics connecting five points."

(3) Professor D. N. LEHMER: "A report on Kulik's manuscript tables of factors."

(4) Professor D. N. LEHMER: "Extension of certain theorems in the theory of quadratic residues."

(5) Professor C. A. NOBLE: "Singular points of a simple kind of a differential equation of the second order."

(6) Professor G. A. MILLER: "Groups in which the subgroup which involves all the substitutions omitting a given letter is regular."

(7) Professor R. E. ALLARDICE: "Note on the cyclide of Dupin."

(8) Dr. J. H. MCDONALD: "On the condition that two circles may have a simultaneously in- and circumscribed quadrilateral."

(9) Professor L. E. DICKSON: "On quadratic forms in a general field."

The papers of Professors Noble, Miller and Dickson were read by title. Abstracts of the papers are given below in order as numbered in the list above.

1. Minkowski has drawn attention to the properties of a type of continued fraction development of a quantity x which is such that if p/q is a convergent, $|p - xq| < 1/2q$, and conversely if $|p - xq| < 1/2q$, p/q is a convergent. Minkowski's method is the use of a geometric representation. It is the object of Dr. McDonald's communication to show how to convert the usual development for x by a simple transformation into this diagonal continued fraction.

2. Professor Blichfeldt proved that a surface of revolution is the only surface possessing the following property: Five arbitrary points on the surface being selected, the ten geodesic distances connecting these points satisfy a relation independent of the coordinates of the five points.

3. Assisted by the Carnegie Institution of Washington, Professor Lehmer has made a careful comparison of his own tables with those previously existing. The first, second and third millions were compared, entry for entry, with Burckhardt's, the fourth, fifth and sixth with Glaisher's, the seventh, eighth and ninth with Dase's. The comparison will be made a second time and a list of errors published.

For the tenth million, Professor Lehmer hoped to avail himself of the manuscript deposited by Rosenberg's widow with the Berlin Academy. Unfortunately, these manuscripts seem to have disappeared. The manuscript tables of Professor Kulik, left in charge of the Vienna Academy, were however available, and with the generous permission of the heirs of Professor Kulik were obtained for examination and comparison.

The Kulik tables are remarkable for their extraordinary extent—to 100,330,201, according to the title page of the first of the six volumes (only one of the volumes was obtained for examination), and for the ingenious table of abbreviations. Primes up to 163 are represented by a single character: $a = 11$, $b = 13$, $c = 17$, etc., to $z = 109$. Then the digits are used (except 7, which was printed into the table as usual): $1 = 113$, $2 = 127$, $\cdots$, $9 = 157$, $0 = 163$. Zero and the

letter o are distinguished in the manuscript by a stroke through the former. From 167 on, the primes are given by double symbols: $aa = 167$, $ba = 173$, etc. By using for the second letter all the letters of the Roman alphabet (j excepted) and all the letters of the German alphabet up to o, the primes are covered as far as 8057.

The arrangement of the table is the same as Burckhardt's. Each page of the manuscript contains 77 columns. The "sieve method" of making the entries was used for primes as high as 1009, after that the "multiple method" was used.

Something like two hundred errors were found in the manuscript for the tenth million. The exact number will be given after the second comparison.

4. Defining a quadratic residue as any number that is congruent to a square, modulo m, and leaving out the usual restriction that the number shall be prime to the modulus, Professor Lehmer obtains in his second paper formulas for the number of distinct residues of this sort.

It is a known theorem that for a prime of the form $4n \pm 1$ there will be found exactly n pairs of quadratic residues that differ by unity. This theorem was also extended to residues as defined above.

5. Professor Noble's paper will appear in full in the BULLETIN.

6. As the groups in which the regular subgroup G_1 is of degree $n-1$, n being the degree of the group G, have received considerable attention, Professor Miller considers mainly those groups in which the degree of G_1 is $n-\alpha$ ($\alpha > 1$). Some of the main results may be stated as follows: G contains n/α systems of imprimitivity which it permutes either according to the group of order 2 or according to a multiply transitive group. When G_1 is also abelian, G contains exactly α^2 substitutions which transform each of these systems into itself, and it contains an additional invariant subgroup of index $n/\alpha - 1$. The latter of these two invariant subgroups includes the former whenever $n/\alpha > 2$. In the special case when $n/\alpha = 2$ this subgroup reduces to identity. To arrive at these results frequent use was made of the following theorems: If a regular group H of degree n is transformed into itself by a substitution s of degree $n-\alpha$ in the same letters, then s is commutative with exactly α of the substitutions of H whenever $\alpha > 0$. When $\alpha = 0$,

s must be commutative with at least one of the substitutions of H besides identity. If a transitive group of degree n is transformed into itself by any substitution in the same letters and if the degree of this substitution is not $n-1$, then it must be commutative with at least one of the substitutions of the transitive group besides identity.

7. The object of Professor Allardice's note on the cyclide of Dupin was to show that, by means of a transformation originally due to Laguerre (see Darboux, Théorie des surfaces, volume 1, page 253), a circle may be transformed into this cyclide; and that the principal properties of the surface may be obtained geometrically by means of the transformation.

8. The relation between the radii and distance between centers giving the condition that two circles may have a simultaneously in- and circumscribed quadrilateral was obtained by various mathematicians (Fuss, Steiner, Jacobi, Cayley) in a form limited to a special case. The complete formulas are found by Dr. McDonald, incidentally giving the interpretation of certain results of the theory of elliptic functions.

9. Professor Dickson's paper appears in full in the present number of the BULLETIN.

W. A. MANNING,
Secretary of the Section.

ON QUADRATIC FORMS IN A GENERAL FIELD.

BY PROFESSOR L. E. DICKSON.

(Read before the San Francisco Section of the American Mathematical Society, September 28, 1907.)

1. WE investigate the equivalence, under linear transformation in a general field F, of two quadratic forms *

$$q \equiv \sum_{i=1}^{n} a_i x_i^2, \quad Q \equiv \sum_{i=1}^{n} \alpha_i X_i^2 \quad (a_i \neq 0,\ \alpha_i \neq 0).$$

An obvious necessary condition is that α_1 shall be representable by q, viz., that there shall exist elements b_i in F such that

$$\alpha_1 = \sum_{i=1}^{n} a_i b_i^2.$$

* Within any field F, not having modulus 2, any quadratic form of non-vanishing determinant is equivalent to one of type q.

We assume that this condition is satisfied. Applying a suitable permutation of the x_i, we may set (§ 2)

$$W_k \equiv \sum_{i=1}^{k} a_i b_i^2 \neq 0 \qquad (k = 1, \cdots, n).$$

In view of § 3, the transformation

$$x_i = b_i y_1 + W_{i-1} y_i - b_i \sum_{j=i+1}^{n} a_j b_j y_j \quad (i = 1, \cdots, n)$$

is of non-vanishing determinant and replaces q by

$$q' \equiv a_1 y_1^2 + \sum_{j=2}^{n} a_j W_j W_{j-1} y_j^2.$$

By the theorem proved in § 4 by a consideration of the automorphs of q', the forms Q and q' (with like first coefficients) are equivalent in F if, and only if,

$$(1) \qquad \sum_{i=2}^{n} \alpha_i X_i^2 = \sum_{j=2}^{n} a_j W_j W_{j-1} y_j^2$$

under a transformation in F on $n-1$ variables. Hence *the necessary and sufficient conditions for the equivalence of the two given n-ary quadratic forms q and Q are that α_1 be representable by q and that the $(n-1)$-ary forms* (1) *be equivalent in F.* The ultimate criteria are that α_1, α_2, $\cdots$, α_n be representable by forms in n, $n-1$, $\cdots$, 1 variables, respectively, whose coefficients are given functions of the a_i. For example, if $n = 2$, the conditions are that α_1 be representable by q and that $a_1 a_2 \alpha_1 \alpha_2$ be a square in F. If $n = 3$, the conditions are that α_1 be representable by q, α_2 by $a_1 a_2 W_2 \xi^2 + a_3 \alpha_1 W_2 \eta^2$, and that $a_1 a_2 a_3 \alpha_1 \alpha_2 \alpha_3$ be a square in F.

2. THEOREM.* *In a non-modular† field F there exists a form*

* *Transactions*, vol. 7 (1906), pp. 276–8. The present proof is decidedly simpler and leads to the explicit expressions (17) for the A_j, as required for the present applications.

† The proof applies to fields having a modulus p, where $p \neq 2$ and $p \geqq n$. It may be extended to apply to any finite field. To q we apply the transformation

$$x_1 = r y_1 + s y_2, \quad x_2 = t y_1 - t^{-1} a_2^{-1} r s a_1 y_2 \qquad (t \neq 0, s \neq 0),$$

and obtain $m y_1^2 + m a_1 s^2 a_2^{-1} t^{-2} y_2^2 + a_3 x_3^2 + \cdots$, where $m = a_1 r^2 + a_2 t^2 \neq 0$. By choice of r and t, we may give m any assigned value in a finite field.

$\sum A_i y_i^2$ equivalent to a given form $q = \sum a_i x_i^2$ and having as its first coefficient A_1 any preassigned mark $\neq 0$ which is representable by q.

By hypothesis, there exist marks b_{i1} of F such that

$$\sum_{i=1}^{n} a_i b_{i1}^2 = A_1 \neq 0. \tag{2}$$

Not every sum of $n-1$ of the terms t_i of (2) vanishes. For if so, we consider the sum lacking t_j and the sum lacking t_k and conclude that $t_j = t_k$ and hence that the n terms are all equal and that each is not zero; but this requires that F shall have a modulus dividing $n-1$. By applying a permutation on the x's in q, we may set $\sum_{i=1}^{n-1} a_i b_{i1}^2 \neq 0$. Not every sum of $n-2$ terms of the latter vanishes, since F does not have a modulus dividing $n-2$, etc. We may therefore set

$$W_k \equiv \sum_{i=1}^{k} a_i b_{i1}^2 \neq 0 \qquad (k = 1, \cdots, n). \tag{3}$$

Under the transformation

$$x_i = \sum_{j=1}^{n} b_{ij} y_j \qquad (i = 1, \cdots, n), \tag{4}$$

q becomes $\sum(A_j y_j^2 + 2B_{jk} y_j y_k)$, $(j, k = 1, \cdots, n\,;\, k > j)$, where

$$A_j = \sum_{i=1}^{n} a_i b_{ij}^2, \quad B_{jk} = \sum_{i=1}^{n} a_i b_{ij} b_{ik}. \tag{5}$$

To make $B_{1k} = 0$, we take

$$b_{1k} = -\, a_1^{-1} b_{11}^{-1} \sum_{i=2}^{n} a_i b_{i1} b_{ik} \qquad (k = 2, \cdots, n). \tag{6}$$

We insert these values in $\Delta = |b_{ij}|$, remove the factor $a_1^{-1} b_{11}^{-1}$ from the first row, then multiply the ith row by $a_i b_{i1}$ and add to the first row, for $i = 2, \cdots, n$. We get

$$\Delta = a_1^{-1} b_{11}^{-1} A_1 \Delta_{11}, \quad \Delta_{11} \equiv |b_{is}| \quad (i, s = 2, \cdots, n). \tag{7}$$

We introduce the abbreviations, in addition to (3),

$$P_{it} = a_i a_t b_{i1} b_{t1}, \quad R_{si} = a_i (a_i b_{i1}^2 + W_{s-1}), \tag{8}$$

so that, in particular, $R_{ss} = a_s W_s$. We find that

$$R_{ss} R_{si} - P_{is}^2 = a_s W_{s-1} R_{s+1,i}, \quad R_{ss} P_{it} - P_{is} P_{st} = a_s W_{s-1} P_{it}. \tag{9}$$

Eliminating the b_{1k} from $a_1 b_{11}^2 B_{jk} (2 \leqq j < k)$ by (6), we get

$$B_{jk}^{(2)} \equiv \sum_{i=2}^{n} b_{ij} \left(R_{2i} b_{ik} + \sum_{t \neq i}^{t=2,\cdots,n} P_{it} b_{tk} \right). \tag{10}$$

Now $R_{22} \equiv a_2 W_2 \neq 0$. To make the coefficient of b_{2j} zero, we take

$$b_{2k} = - R_{22}^{-1} \sum_{t=3}^{n} P_{2t} b_{tk} \qquad (k = 3, \cdots, n). \tag{11}$$

Set $b_{i2} = 0$ $(i > 2)$. Then $B_{2k}^{(2)}$ evidently vanishes, while

$$\Delta_{11} = b_{22} \Delta_{22}, \quad \Delta_{22} \equiv |b_{is}| \qquad (i, s = 3, \cdots, n). \tag{12}$$

Generalizing (6) and (11), we shall take

$$b_{sk} = - R_{ss}^{-1} \sum_{t=s+1}^{n} P_{st} b_{tk} \quad (k = s+1, \cdots, n), \tag{13}$$

$$b_{il} = 0 \qquad (i > l > 1), \tag{14}$$

and then prove by induction from s to $s+1$ that, for

$$1 < s \leqq j < k \leqq n,$$

the product of B_{jk} by a non-vanishing factor equals

$$B_{jk}^{(s)} \equiv \sum_{i=s}^{n} b_{ij} \left(R_{si} b_{ik} + \sum_{t \neq i}^{t=s,\cdots,n} P_{it} b_{tk} \right). \tag{15}$$

By (10) this statement is true for $s = 2$. The coefficient of b_{sj} in (15) is zero by (13), so that we may set $i > s$. From the term given by $t = s$, we eliminate b_{sk} by means of (13). Hence

$$R_{ss} B_{jk}^{(s)} = \sum_{i=s+1}^{n} b_{ij} \left\{ (R_{ss} R_{si} - P_{is}^2) b_{ik} + \sum_{t \neq i}^{t=s+1,\cdots,n} (R_{ss} P_{it} - P_{is} P_{st}) b_{tk} \right\}.$$

Applying (9), we get

$$R_{ss} B_{jk}^{(s)} = a_s W_{s-1} B_{jk}^{(s+1)},$$

so that the induction is complete. Thus for given values of j and k, $1 < j < k$, we may increase s and make $s > j$; then (15) vanishes by (14). Hence every $B_{jk} = 0$ $(k > j)$. Finally, if

we take each $b_{ii} \neq 0$, the determinant of (4) is not zero, in view of (14) and (7) or (12).

3. From (13), in combination with (14) and (9_2), we get

$$b_{sk} = -\alpha_s^{-1} W_{k-1}^{-1} P_{sk} b_{kk} \qquad (s < k). \tag{16}$$

Starting with the simple formulas (14) and (16), we may readily verify that, in (5), each $B_{jk} = 0$, and

$$A_j = a_j W_j W_{j-1}^{-1} b_{jj}^2 \qquad (j = 2, \cdots, n). \tag{17}$$

THEOREM. *For given elements b_{ii} satisfying* (2) *and* (3), *and any elements $b_{jj} \neq 0$ $(j > 1)$, the transformation, of non-vanishing determinant,*

$$x_i = b_{ii} y_1 + b_{ii} y_i - b_{ii} \sum_{j=i+1}^{n} a_j b_{j1} b_{jj} W_{j-1}^{-1} y_j \quad (i = 1, \cdots, n) \tag{18}$$

replaces $\sum a_i x_i^2$ by $\sum A_j y_j^2$, the A_j being given by (2), (17).

If we employ the special values $b_{jj} = W_{j-1}$ and set $b_{11} = b_1$, we obtain the simpler results given in § 1.

4. Within the field F, let the forms

$$Q = \sum_{i=1}^{n} \alpha_i X_i^2, \quad E = \sum_{i=1}^{n} e_i y_i^2 \quad (\alpha_i \neq 0,\ e_i \neq 0) \tag{19}$$

be equivalent under the transformation

$$S: \quad X_i = \sum_{j=1}^{n} \sigma_{ij} y_j \qquad (i = 1, \cdots, n). \tag{20}$$

In view of the formulas

$$2 e_i y_i = \frac{\partial E}{\partial y_i} = \frac{\partial Q}{\partial y_i} = \sum_{j=1}^{n} 2 \alpha_j X_j \frac{\partial X_j}{\partial y_i} = 2 \sum_{j=1}^{n} \alpha_j \sigma_{ji} X_j,$$

the inverse of S is

$$S^{-1}: \quad y_i = e_i^{-1} \sum_{j=1}^{n} \alpha_j \sigma_{ji} X_j \qquad (i = 1, \cdots, n). \tag{21}$$

Eliminating the y_i from $Q = E$, we get

$$\sum_{i=1}^{n} e_i^{-1} \sigma_{ji}^2 = \alpha_j^{-1}, \quad \sum_{i=1}^{n} e_i^{-1} \sigma_{ji} \sigma_{ki} = 0 \qquad (j \neq k). \tag{22}$$

Next, let $E(y) = mE(\xi)$ under the transformation

$$C: \quad \xi_i = \sum_{j=1}^{n} \gamma_{ij} y_j \qquad (i = 1, \cdots, n). \tag{23}$$

Replacing α_j by me_j and σ_{ji} by γ_{ji} in (21) and (22), we get

$$C^{-1}: \quad y_i = me_i^{-1} \sum_{j=1}^{n} e_j \gamma_{ji} \xi_j \qquad (i = 1, \cdots, n), \tag{24}$$

$$\sum_{i=1}^{n} e_i^{-1} \gamma_{ji}^2 = m^{-1} e_j^{-1}, \quad \sum_{i=1}^{n} e_i^{-1} \gamma_{ji} \gamma_{ki} = 0 \qquad (j \neq k). \tag{25}$$

Eliminating the y_i between (20) and (24), we obtain relations of the form

$$X_i = \sum_{j=1}^{n} \beta_{ij} \xi_j \qquad (i = 1, \cdots, n), \tag{26}$$

which imply $Q = mE(\xi)$. We desire that the first of these relations shall reduce to $X_1 = \xi_1$. By (20), (23), the conditions are

$$\gamma_{1j} = \sigma_{1j} \qquad (j = 1, \cdots, n). \tag{27}$$

In view of (22_1) and (25_1), for $j = 1$, these require that $me_1 = \alpha_1$. Conversely, if $Q = E$ under S, and if there exists in F a matrix (γ_{ij}) with the first row identical with that of S and such that $E(y) = e_1^{-1}\alpha_1 E(\xi)$ under C, then will $Q = e_1^{-1}\alpha_1 E(\xi)$ under a transformation (26) with $X_1 = \xi_1$, and therefore

$$\sum_{i=2}^{n} \alpha_i X_i^2 = e_1^{-1}\alpha_1 \sum_{i=2}^{n} e_i \xi_i^2, \tag{28}$$

under a transformation in F on $n - 1$ variables.*

As noted in § 1, it suffices to treat the case † $e_1 = \alpha_1$. We have therefore to deal with automorphs of $E(y)$. In view of

* For $\sigma_{1i} = 0$ $(i > 1)$, (22_2) with $j = 1$ gives $\sigma_{k1} = 0$ $(k > 1)$.

† The general case presents an essential difficulty. By (24), $|C^{-1}| = m^n |C|$, so that $m^n = \Delta^{-2}$, where $\Delta = |\gamma_{ij}|$. For n odd, m must therefore be a square in F, and hence cannot, in general, be made equal to $e_1^{-1}\alpha_1$, where α_1^{-1} is an arbitrary element of the form $\sum e_i^{-1}\sigma_{1i}^2$; see (22_1) for $j = 1$. The case $n = 2$ is quite simple; we may take as (γ_{ij}) the matrix

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ -e_2^{-1}e_1\sigma_{12} & \sigma_{11} \end{pmatrix}, \quad e_1^{-1}\sigma_{11}^2 + e_2^{-1}\sigma_{12}^2 = \alpha_1^{-1}.$$

§ 5, we can determine a matrix (γ_{ij}) satisfying the above conditions. We therefore have the

THEOREM. *If Q and E are equivalent in F under S and if $e_1 = \alpha_1$, then Q is equivalent to $E(\xi)$ under a transformation in F with $X_1 = \xi_1$, so that $\sum_{i=2}^{n}\alpha_i X_i^2$ and $\sum_{i=2}^{n}e_i\xi_i^2$ are equivalent in F under a transformation on $n-1$ variables.*

Conversely, if the latter forms are equivalent in F, then evidently Q and E, with $e_1 = \alpha_1$, are equivalent in F.

5. THEOREM. *For any set of solutions σ_i in F of*

$$\sum_{i=1}^{n} e_i^{-1}\sigma_i^2 = e_1^{-1}, \tag{29}$$

the quadratic form $E = \sum e_i y_i^2$ has an automorph in F which replaces y_1 by $\sum \sigma_j y_j$.

Let Y be any skew symmetric matrix, I the identity (unit) matrix, E^{-1} the inverse of the matrix of E. Let $Z = E^{-1}Y$. Then, by Cayley's theorem, E has the automorph

$$P \equiv (I+Z)^{-1}(I-Z) \quad (\Delta \equiv |I+Z| \neq 0).$$

It will suffice to take as Y the skew matrix in which the elements outside of the first row and column are all zero, while the first row is

$$0, c_2, \cdots, c_n.$$

Hence

$$I \pm Z = \begin{pmatrix} 1 & \pm c_2 e_1^{-1} & \pm c_3 e_1^{-1} & \cdots & \pm c_n e_1^{-1} \\ \mp c_2 e_2^{-1} & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \mp c_n e_n^{-1} & 0 & 0 & \cdots & 1 \end{pmatrix},$$

$$\Delta = 1 + \kappa, \quad \kappa \equiv e_1^{-1}\sum_{i=2}^{n} e_i^{-1}c_i^2. \tag{30}$$

The first row of $(I+Z)^{-1}$ is

$$\frac{1}{\Delta}, \quad \frac{-c_2}{e_1\Delta}, \quad \frac{-c_3}{e_1\Delta}, \quad \cdots, \quad \frac{-c_n}{e_1\Delta}.$$

Hence the first row of the product P is

$$\frac{1-\kappa}{\Delta}, \quad \frac{-2c_2}{e_1\Delta}, \quad \frac{-2c_3}{e_1\Delta}, \quad \cdots, \quad \frac{-2c_n}{e_1\Delta}.$$

These elements are to be made equal to

$$\sigma_1,\ \sigma_2,\ \sigma_3,\ \cdots,\ \sigma_n,$$

respectively. Hence, by (30_1), we take

$$2=\Delta(1+\sigma_1),\quad \kappa=\tfrac{1}{2}\Delta(1-\sigma_1),\quad c_i=-\tfrac{1}{2}\Delta e_1\sigma_i\quad (i=2,\cdots,n).$$

Eliminating the c_i from (30_2), and applying (29), we get

$$\kappa=\tfrac{1}{4}\Delta^2 e_1\sum_{i=2}^{n}e_i^{-1}\sigma_i^2=\tfrac{1}{4}\Delta^2(1-\sigma_1^2).$$

Hence the conditions may be satisfied if $1+\sigma_1\neq 0$. But for $\sigma_1=-1$, we may first apply the automorph $y_1'=-y_1$.

6. Although not employed in the present paper, the following generalization of the preceding theorem may be noted:

THEOREM. *Let r be any positive integer $\leqq n$. If the σ_{ji} are any solutions in F of relations* (22), *for $j,k=1,\cdots,r$, and $a_j=e_j$, the form E has an automorph in F which replaces y_i by $\sum_{j=1}^{n}\sigma_{ij}y_j$, for $i=1,\cdots,r$.*

The proof by induction, based on the result of § 5, is similar to that in the *American Journal*, volume 23 (1901), page 344, for the special case of a finite field with special values of the e_i.

7. Let F be field R of all rational numbers. There exist* rational values of $b_1,\cdots,b_4$ such that $\sum_{i=1}^{4}a_ib_i^2$ equals $+1$ or -1, according as $a_1,\cdots,a_4$ are not all negative, or all negative. Hence, by § 1, any n-ary rational quadratic form of non-vanishing determinant is reducible by a linear transformation with rational coefficients to one of the forms

$$f_{p,a,b,c}\equiv\sum_{i=1}^{p}x_i^2-\sum_{i=p+1}^{n-3}x_i^2+ax_{n-2}^2+bx_{n-1}^2+cx_n^2,$$

in which a, b, c are all negative if $p<n-3$, while $f_{p,a,b,c}$ is reducible to $f_{p,\alpha,\beta,\gamma}$ if, and only if,† the ternary form

$$t_{a,b,c}\equiv ax^2+by^2+cz^2$$

is reducible in R to $t_{\alpha,\beta,\gamma}$ (see end of § 1).

* A simple consequence (*Transactions*, l. c., p. 279) of a theorem due to A. Meyer; cf., Bachmann, Zahlentheorie, IV_1, p. 266.

† This part of the result was not given in my former paper.

ON THE CANONICAL SUBSTITUTION IN THE HAMILTON-JACOBI CANONICAL SYSTEM OF DIFFERENTIAL EQUATIONS.

BY DR. D. C. GILLESPIE.

(Read before the American Mathematical Society, September 5, 1907.)

LIE established a one-to-one correspondence between the integrals of the canonical system of differential equations and the one-parameter continuous groups of contact transformations of which the system admits, *i. e.*, making use of an integral of the system one can construct the infinitesimal transformation of a group of which the system admits, or, on the other hand by making use of the infinitesimal transformation of a group of which the system admits one can construct an integral of the system.* This theorem is the foundation of the modern transformation theory of dynamical systems.† The single canonical substitution (introduced by Jacobi) is of importance in the transformation and simplification of the dynamical equations.‡

The purpose of this paper is to define a type of the single canonical substitution which leads to an integral of the equations, *i. e.*, if one knows a member of the defined type of canonical substitutions one can construct an integral of the system using only algebraic operations.

A system of differential equations which has the form

$$(1)\quad \frac{d\pi}{dx}=\frac{\partial H}{\partial y},\quad \frac{dy}{dx}=-\frac{\partial H}{\partial \pi},\quad \frac{d\kappa}{dx}=\frac{\partial H}{\partial z},\quad \frac{dz}{dx}=-\frac{\partial H}{\partial \kappa}$$

is called a canonical system. H is a known function of y, π, z, κ and x; y, π, z and κ are the unknown functions; thus the solution of the system of four equations (1) consists in determining the four unknowns y, π, z and κ as such functions of x that the equations become identities in x. The functions y and π, as also z and κ, are called conjugate.

A substitution which leaves the form of the system (1) unchanged, though the function H may or may not be changed, is a canonical substitution.

* Whittaker, Analytical Dynamics, page 308.
† Ibid., page 292.
‡ Poincaré, Mécanique céleste; Jacobi, Vorlesungen über Dynamik.

In order to avoid complicated formulas I take a system of four equations. The generalization of the theory from four to $2n$ equations is evident.

The system of equations (1) expresses the necessary condition that the integral

$$\int_1^2 \{\pi y' + \kappa z' + H(y, \pi, z, \kappa, x)\}dx \tag{2}$$

(where $dy/dx \equiv y'$, $dz/dx \equiv z'$, and 1 and 2 stand for any limits of integration) take a maximum or a minimum value; *i. e.*, if the question be, what functions of x must y, π, z and κ be in order that the integral (2) shall take an extreme value, the answer is, only such functions as satisfy the equations (1).

In order that y and z be such functions of x that the integral

$$\int_1^2 F(y', y, z', z, x)dx \tag{3}$$

take an extreme value, it is necessary that y and z satisfy the system of equations

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0, \quad \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right) - \frac{\partial F}{\partial z} = 0. \tag{4}$$

If now

$$F(y', y, z', z, x) \equiv H(y, \pi, z, \kappa, x) + y'\pi + z'\kappa, \tag{5}$$

whereby is understood that instead of π and κ are substituted their values in terms of y', y, z', and z, determined from the equations

$$y' = -\frac{\partial H(y, \pi, z, \kappa, x)}{\partial \pi}, \quad z' = -\frac{\partial H(y \cdot \pi, z, \kappa, x)}{\partial \kappa}, \tag{6}$$

then the two systems (4) and (1) are equivalent. In fact let

$$\begin{aligned} y &= y(x, c_1, c_2, c_3, c_4), \quad z = z(x, c_1, c_2, c_3, c_4), \\ \pi &= \pi(x, c_1, c_2, c_3, c_4), \quad \kappa = \kappa(x, c_1, c_2, c_3, c_4) \end{aligned} \tag{7}$$

(c_1, c_2, c_3, c_4 are arbitrary constants) be the complete solution of (1), then

$$y = y(x, c_1, c_2, c_3, c_4), \quad z = z(x, c_1, c_2, c_3, c_4) \tag{8}$$

is the complete solution of (4). Both systems of equations are satisfied by the same values of y and z. Conversely now let

$$y = y(x, c_1, c_2, c_3, c_4), \quad z = z(x, c_1, c_2, c_3, c_4)$$

be the complete solution of (1), then

$$y = y(x, c_1, c_2, c_3, c_4), \quad z = z(x, c_1, c_2, c_3, c_4),$$

$$(9) \quad \pi = \frac{\partial}{\partial y'} F(y', y, z', z, x), \quad \kappa = \frac{\partial}{\partial z'} F(y', y, z', z, x)$$

is the complete solution of (4). Thus the problems of integrating these two systems of equations are equivalent.

We will now suppose a canonical substitution known for the system of differential equations (4),

$$(10) \quad \pi_1 = f_1(y, \pi, z, \kappa, x), \ \kappa_1 = f_2(y, \pi, z, \kappa, x), \ z_1 = f_3(y, \pi, z, \kappa, x),$$

$$y_1 = f_4(y, \pi, z, \kappa, x), \quad x_1 = f_5(y, \pi, z, \kappa, x).$$

Under this substitution the $H(y, \pi, z, \kappa, x)$ goes over into another function $H^*(y_1, \pi_1, z_1, \kappa_1, x_1)$, but the form of the equations remains canonical. We wish to impose a condition that will fix the substitution (10) as a member of the type of canonical substitutions which leads to an integral of (4). The condition is that the y and z functions of the independent variable x remain unchanged. This means that if

$$y = \phi_1(x, c_1, c_2, c_3, c_4), \quad z = \phi_2(x, c_1, c_2, c_3, c_4),$$

$$\pi = \phi_3(x, c_1, c_2, c_3, c_4), \quad \kappa = \phi_4(x, c_1, c_2, c_3, c_4)$$

is the complete solution of (4), then

$$y_1 = \phi_1(x_1, c_1, c_2, c_3, c_4), \quad z_1 = \phi_2(x_1, c_1, c_2, c_3, c_4),$$

$$\pi_1 = \omega_1(x_1, c_1, c_2, c_3, c_4), \quad \kappa_1 = \omega_2(x_1, c_1, c_2, c_3, c_4)$$

(where ω_1 and ω_2 mean simply functions different from ϕ_3 and ϕ_4) is the complete solution of the system of equations obtained by subjecting the system (1) to the substitution (10).

In order to determine whether a given canonical substitution belongs to the defined type, it would seem necessary to know

the complete solutions both of the system of equations (1) and the system obtained by subjecting (1) to the given substitution. This however is not the case. For suppose again that (10) is any given canonical substitution of which we wish to determine whether it belongs to the defined type. Transforming the system of equations (1) by (10), we obtain a new function

$$H^*(y_1, \pi_1, z_1, \kappa_1, x)$$

and hence a new system of differential equations

$$(11) \qquad \begin{aligned} \frac{d\pi_1}{dx_1} &= \frac{\partial H^*}{\partial y_1}, & \frac{d\kappa_1}{dx_1} &= \frac{\partial H^*}{\partial z_1}, \\ \frac{dy_1}{dx_1} &= -\frac{\partial H^*}{\partial \pi_1}, & \frac{dz_1}{dx_1} &= -\frac{\partial H^*}{\partial \kappa_1}. \end{aligned}$$

Making use of this new function $H^*(y_1, \pi_1, z_1, \kappa_1, x_1)$, we form a new function $F^*(y_1', y_1, z_1', z_1, x_1)$, just as in (5),

$$F^*(y_1', y_1, z_1', z_1, x) \equiv H^*(y_1, \pi_1, z_1, \kappa_1, x_1) + y_1'\pi_1 + z_1'\kappa,$$

where instead of π_1 and κ_1 are substituted their values in terms of y_1', y_1, z_1' and z_1 determined from the equations

$$y_1' = -\frac{\partial H^*(y_1, \pi_1, z_1, \kappa_1, x_1)}{\partial \pi_1}, \quad z_1' = -\frac{\partial H^*(y_1, \pi_1, z_1, \kappa_1, x_1)}{\partial \kappa_1}.$$

We express now the necessary condition that the integral

$$(12) \qquad \int_1^2 F^*(y_1', y_1, z_1', z_1, x_1)dx_1$$

take an extreme value and obtain the equations

$$(13) \qquad \frac{d}{dx_1}\left(\frac{\partial F^*}{\partial y_1'}\right) - \frac{\partial F^*}{\partial y_1} = 0, \quad \frac{d}{dx_1}\left(\frac{\partial F^*}{\partial z_1'}\right) - \frac{\partial F^*}{\partial z_1} = 0.$$

The systems of equations (1) and (4) are satisfied by the same functional values of y and z, $y = f_1(x, c_1, c_2, c_3, c_4)$, $z = f_2(x, c_1, c_2, c_3, c_4)$; so also are the systems (11) and (13), $y_1 = f_1^*(x_1, c_1, c_2, c_3, c_4)$, $z_1 = f_2^*(x_1, c_1, c_2, c_3, c_4)$. If now the transformation (10) belongs to the defined type, then $f_1(x, c_1, c_2, c_3, c_4) \equiv f_1^*(x, c_1, c_2, c_3, c_4)$ and

$f_2(x, c_1, c_2, c_3, c_4) \equiv f_2^*(x, c_1, c_2, c_3, c_4)$. This means of course that if we drop the subscripts from x_1, y_1, z_1 in the system (13), it becomes identical with (4). This then is the condition that a given transformation be of the defined type.

Since the system of equations (4) expresses the necessary condition that both the integrals

$$\int_1^2 F(y', y, z', z, x)dx, \qquad \int_1^2 F^*(y', y, z', z, x)dx$$

take an extreme value,*

$$\begin{vmatrix} F_{y'y'} & F_{y'z'} \\ F_{z'y'} & F_{z'z'} \end{vmatrix} \div \begin{vmatrix} F^*_{y'y'} & F^*_{y'z'} \\ F^*_{z'y'} & F^*_{z'z'} \end{vmatrix} = \text{const.} \left(F_{ww'} \equiv \frac{\partial^2 F}{\partial y^2}, \text{ etc.} \right)$$

is an integral of the system (4). This, as we have seen, is equivalent to the construction of an integral of the system (1).

Example. As a simple example where the systems consist of only two equations, we take

$$\frac{d\pi}{dx} = 0, \quad \frac{dy}{dx} = -\pi; \tag{1}$$

here $H \equiv \frac{1}{2}\pi^2$, $F \equiv H + y'\pi \equiv \frac{1}{2}\pi^2 + y'\pi$, or since by (1) $y' = -\pi$, $F \equiv -\frac{1}{2}y'^2$. The integral $\int_1^2 -\frac{1}{2}y'dx$ has as necessary condition for an extreme value

$$y'' = 0. \tag{2}$$

We subject (1) to the substitution

$$x = y_1, \quad y = -x_1, \quad \pi = \pi_1, \tag{3}$$

and obtain

$$\frac{d\pi_1}{dx_1} = 0, \quad \frac{dy_1}{dx_1} = -\frac{1}{\pi_1}, \tag{4}$$

$H^* = \log \pi_1, F^* = \log \pi_1 + y'\pi_1 = \log(-1/y_1') - 1$, the necessary

*D. C. Gillespie, "On the construction of an integral of Lagrange's equations in the calculus of variations," BULLETIN, 2d series, vol. 13, no. 7, pp. 345–348.

condition that the integral $\int_1^2 (\log(-1/y_1') - 1)\,dx$ take an extreme value is

$$y_1'' = 0, \tag{5}$$

the substitution (3) is a member of the defined type and $F_{y'y'} \div F_{yy'} \equiv y' = \text{const.}$ is an integral of (2). $y' = c$, $\pi = \partial F/\partial y' = -y'$ gives $\pi = -c$ for an integral of (1).

CORNELL UNIVERSITY,
August, 1907.

THE MAXIMUM VALUE OF A DETERMINANT.

BY DR. F. R. SHARPE.

HADAMARD* has shown that the maximum value of a determinant when the absolute value of each element does not exceed 1 is $n^{\frac{1}{2}n}$. The square of such a maximum determinant is a determinant having all its elements 0 except those of the principal diagonal. If the elements are restricted to real values, they are each ± 1 and are so arranged that when compared row with row there is always an equal number of changes and permanences of sign amongst the corresponding elements. Hence n is necessarily even. If we compare any two rows with a third row, the division of changes and permanences is again even. Hence n must be a multiple of 4. By a rearrangement of signs and order of columns we can always arrange any three rows in the form which for the case of $n = 12$ is

1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	−1	−1	−1	−1	−1	−1
1	1	1	−1	−1	−1	1	1	1	−1	−1	−1

The actual maximum determinant is known for the following cases: (1) n a power of 2, (2) $n = 12$ or 20, (3) when the factors of n are any of the preceding numbers. For example, when n is 8, the determinant is

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* *Bull. des Sciences math.*, 1893.

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{vmatrix} = 4096.$$

The object of this note is to determine the probable maximum value which the determinant actually attains when n is not a multiple of 4, and to prove certain properties of a determinant which does attain the maximum value $n^{\frac{1}{2}n}$ for real values of the elements.

THEOREM I. *The cofactor of any element of a maximum determinant of order* $4p$ *and value* $(4p)^{2p}$ *is* $\pm (4p)^{2p-1}$, *the sign being the same as the sign of that element.*

This follows at once from the form of the square of the determinant and the fact that if we combine the cofactors of one row with the elements of any other row the result is 0.

THEOREM II. *The second minors of a maximum determinant of order* $4p$ *and value* $(4p)^{2p}$ *are* $\pm 2(4p)^{2p-2}$ *or* 0, *according as the complementary minor is* 2 *or* 0.

Arrange the determinant as explained above and consider the minor of the first element of the first row. The cofactors of its first row are clearly 0 for the first $2p - 1$ elements and $2(4p)^{2p-2}$ for the last $2p$ elements, because of the equal distribution of the signs of the elements of the last $2p$ columns.

THEOREM III. *The third minors of a maximum determinant of order* $4p$ *and value* $(4p)^{2p}$ *are* $\pm 4(4p)^{2p-3}$ *or* 0, *according as the complementary minor is* 4 *or* 0.

The determinant being arranged as in the previous theorem, consider the second minor obtained by omitting the first and second rows and the first and $(2p + 1)$th columns. The cofactors of its first row are clearly 0 for the first $p - 1$ elements, $-4(4p)^{2p-3}$ for the next p elements and the last p elements, and 0 for the remaining $p - 1$ elements, because of the equal distribution of signs among the second p and last p columns of the original determinant.

THEOREM IV. *If the sign of any element of a maximum determinant of order* $4p$ *and value* $(4p)^{2p}$ *is changed and its row added to the original determinant together with a suitably chosen column, a determinant of order* $4p+1$ *is formed with a value* $(16p-4)(4p)^{2p-1}$.

Expand the new determinant in terms of the elements of the new column, the signs of which are to be chosen so that the the $4p+1$ determinants obtained are all positive. We obtain the original determinant, the original determinant with a change of sign in one element, and $4p-1$ other determinants with two rows identical except in the sign of one element. Hence the new determinant has the value

$$(4p)^{2p}+(4p)^{2p}-2(4p)^{2p-1}+2(4p-1)(4p)^{2p-1}=(16p-4)(4p)^{2p-1}.$$

Using Theorems I, II and IV, we see that at any rate for $p=1, 2, 3, 4,$ or 5 determinants exist

of order $4p-2$, value $2(4p)^{2p-2}$;
of order $4p-1$, value $(4p)^{2p-1}$;
of order $4p$, value $(4p)^{2p}$;
of order $4p+1$, value $(16p-4)(4p)^{2p-1}$.

These give for

$$n=2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\cdots,\ 16$$

values 2, 4, 16, 48, 128, 512, 4096, ..., 4 294 967 296.

Professor E. W. Davis's formula * $(n-2)2^{n-1}$ gives for

$$n=3,\ 4,\ 5,\ 6,\ 7,\ 8,\cdots,\ 16$$

values 4, 16, 48, 128, 320, 768, ..., 458 752.

This formula is clearly too small for $n>6$; in fact, when n is a power of 2 above 5, Hadamard's value is greater than Davis's for the next higher power of 2, and the former's value for $n=24$ is larger than the latter's for $n=2^5$.

CORNELL UNIVERSITY,
October, 1907.

* BULLETIN, October, 1907.

THIRD REPORT ON RECENT PROGRESS IN THE THEORY OF GROUPS OF FINITE ORDER.

BY PROFESSOR G. A. MILLER.

(*Continued from page* 91.)

§ 3. Group of Isomorphisms.

In view of the fundamental importance of the group of isomorphisms of any group it is desirable to have theorems by means of which the group of isomorphisms I of a given substitution group G can be readily determined. If any G of degree n contains n distinct subgroups of degree $n-1$ which are composed of all the substitutions omitting fixed letters, then its substitutions transform these n subgroups just as they transform their own letters and hence G contains no substitution besides identity which is commutative with each one of these n subgroups of degree $n-1$. When G is transitive it will have n such subgroups, provided it has one subgroup of degree $n-1$. Any operator which transforms each of these n subgroups into itself must therefore be commutative with every operator of G. If G does not involve a subgroup of degree n which may correspond to such a subgroup of degree $n-1$ in a holomorphism of G, then its I can be represented as a substitution group of degree n which involves G as an invariant subgroup.

If G is a transitive group and contains a subgroup of degree $n-1$, its I may be represented as a transitive substitution group whose elements are the subgroups which may correspond to one of the largest subgroups of degree $n-1$ in a holomorphism of G. If the degree of this transitive group exceeds n, it must be imprimitive and the m conjugate largest subgroups of degree $n-1$ constitute one system of imprimitivity, while its other systems correspond to subgroups of degree and of index n under G. While these general theorems are frequently directly useful to determine the I of a given G, a number of recent more special theorems find wide application. Among these are the following:

If an abelian group G which involves operators whose orders exceed 2 is extended by means of an operator of order 2 which transforms each operator of G into its inverse, then the I of

this extended group is the holomorph of G.* The I of the group obtained by extending the cyclic group of order $2m$ ($m > 2$) by means of an operator of order 4 which transforms each of its operators into its inverse is the holomorph of this cyclic group. The square of a complete group has the double holomorph of this group for its I. The necessary and sufficient condition that a holomorphism corresponds to an invariant operator under I is that the operators which correspond to themselves form an invariant subgroup and that the remaining operators correspond to themselves multiplied by invariant operators.†

If an operator of order 2 in I transforms an operator s of G into $s_1 s$, then it transforms s_1 into its inverse. Hence every operator of order 2 in I transforms some operators of G into their inverses. Moreover, if such an operator transforms every operator of G except identity into a different operator, it must transform every operator of G into its inverse and hence G is an abelian group of odd order. Burnside considered the properties of G when I contains operators of order 3 which transform every operator of G except identity into a different operator. It may be remarked that the review of this note in the *Jahrbuch über die Fortschritte der Mathematik*‡ is misleading, since it does not state that the operator of order 3 under consideration transforms all of the operators of G except identity into different operators.

By means of the preceding theorems it is easy to find the groups of isomorphisms of substitution groups of low degrees. This has been done for all the groups which can be represented on 7 or a smaller number of letters, as well as for the simple groups whose degrees do not exceed 14. Among the latter the group of order 7920 is especially interesting since it is both complete and simple. It is not difficult to see that the direct product of a complete group which contains only one subgroup of index 2 and the group of order 2 is simply isomorphic with its I. In particular, the direct product of the symmetric group whose degree is not 2 or 6 and the group of order 2 is simply isomorphic with its I.

The groups of isomorphisms of a number of special types of

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* *Amer. Jour. of Mathematics*, vol. 29 (1907), p. 4.
† *Transactions Amer. Math. Society*, vol. 4 (1903), p. 153.
‡ Vol. 34 (1905), p. 160.

groups of order p^m have recently been determined. Young * determined these groups for such groups of order p^m as contain cyclic subgroups of order p^{m-1}. In chapter 2 of Le Vavasseur's work cited above, he considers groups of isomorphisms of the groups of order p^4, and in chapter 3 he considers the groups of cogredient isomorphisms of a large number of the groups of order 2^5. In the current volume of the *Transactions* of this Society, Ranum generalizes the linear congruence groups of Jordan "by using different moduli for the different elements of the matrices, so that each element is a residue of its own particular modulus." He applies these generalized linear congruence groups to the group of isomorphisms of any abelian group and obtains a number of important new theorems. One of these affirms that the necessary and sufficient condition that the group of isomorphisms of an abelian group of order p^m be solvable is that all its invariants are distinct when $p > 3$, and that no three of the invariants are equal to each other when $p = 2$ or 3.†

The group of isomorphisms of every finite group is finite since its order cannot exceed $(g-1)!$, g being the order of the group. There are only four groups for which the order of I has this maximal value; viz., the groups of orders, 1, 2, 3 and the non-cyclic group of order 4. As a rule the order of I is very much smaller, since the possible holomorphisms are greatly restricted by the properties of the operators. It has been observed that in a non-abelian group not more than three-fourths of the operators may correspond to their inverses and the groups which have this property have been considered. From this fact it follows that an abelian group may be defined by the property that more than three-fourths of its operators may correspond to their inverses in a holomorphism of the group.‡ Manning has considered the groups in which five-eighths or more of the operators may correspond to their inverses and in this connection proved a fundamental theorem in regard to the properties of groups involving two invariant subgroups which have only identity in common.§

The group of isomorphisms of the cyclic group of order $2\,3^m$ is the cyclic group of order $2 \cdot 3^{m-1}$. Hence it follows

* Young, *Amer. Jour. of Mathematics*, vol. 25 (1903), p. 206.
† Ranum, *Transactions Amer. Math. Society*, vol. 8 (1907), p. 89.
‡ *Annals of Mathematics*, vol. 7 (1906), p. 59.
§ Manning, *Transactions Amer. Math. Society*, vol. 7 (1906), p. 223.

that each of the m successive groups of isomorphisms of this cyclic group as well as of the cyclic group of order 3^m is cyclic while the $(m+1)$th is identity. The only other groups which have the property that all their successive groups of isomorphisms are cyclic are the group of order 4 and those whose order is a prime number of the form $2 \cdot 3^m + 1$. From the fact that the order of the group of isomorphisms of every group whose order exceeds 2 is greater than unity it follows that we arrive at an infinite system of groups by forming the successive holomorphs of any group whose order exceeds 2. The characteristic operators of a group are the invariant operators of its holomorph. Since *an abelian group cannot have more than one characteristic operator besides identity*, it results that each of the successive holomorphs of an abelian group has either one or no invariant operator besides identity, as the abelian group has one or no characteristic operator in addition to identity. The question whether a non-abelian group can have more than one characteristic operator besides identity remains unsettled. This is also true of the question whether a non-abelian group can have an abelian group of isomorphisms. If such a group exists there must be a metabelian group of order p^a which has the same property.

The characteristic properties of a complete group are that it does not admit outer isomorphisms and none of its operators except identity is invariant under the group. A large number of well-known groups have the latter property without having also the former. It seems desirable to find groups which have the former property without having also the latter. In fact, the second part of the definition can only be justified by a proof of the existence of such a group. This proof is included in the proof that there exists a group of composite order which is both simple and complete since the direct product of such a group and the group of order 2 admits no outer isomorphisms but includes an invariant operator of order 2.* In view of the historic interest in the five-fold transitive groups of degrees 12 and 24 respectively it is of special interest to note that the latter is a complete group while the former has a group of twice its own order for its group of isomorphisms.

* *Messenger of Mathematics*, vol. 37 (1907), p. 54.

§ 4. Substitution Groups.

In 1896 "Le grand prix des sciences mathématiques" was awarded to Maillet for his memoir entitled "Recherches sur la classe et l'ordre des groupes de substitutions." The memoir was not published until about six years afterwards * and hence its publication comes within the period covered by the present report. The memoir covers 120 pages and is divided into two parts. The first is devoted to the class of the primitive substitution groups which are simply isomorphic with either a symmetric or an alternating group, and it is followed by two notes in which hypersystems and primitive groups whose class is not less than four-fifth of the degree are considered. The second part, which the author considers the more important, is devoted to the limit of the order of the groups of degree n which do not involve the alternating group of this degree.

The term hypersystems is introduced with a view to exhibiting more clearly the connection between substitution groups and Lie's transformation groups, as well as to generalize some results previously published. Most of the results are extensions of theorems due to Jordan and Bochert. Some of these relate to the class of a primitive group which does not include the alternating group of the same degree. Manning considers the same question in several recent memoirs. † In most of the older work along this line the degree of such a group was assumed to be given and the smallest possible class of the primitive group was considered, while Manning assumes the minimum class and finds the maximum degree for this class.

The important theorem that every substitution group of prime degree p which contains more than one subgroup of order p is at least doubly transitive was first proved by means of group characteristics. Burnside has recently given a much simpler proof based upon a purely arithmetical property of the prime roots of unity. ‡ The enumeration of all the possible substitution groups of a given degree has been materially advanced during the period under consideration. Miss Martin's enumeration of the imprimitive groups of degree 15 was completed by the list published by Kuhn. This list is preceded

* Maillet, *Mémoires présentés par divers Savants a l' Académie des Sciences de l' Institut national de France*, vol. 32 (1902).

† Manning, *Transactions Amer. Math. Society*, vol. 4 (1903), p. 351; vol. 6 (1905), p. 42; Bulletin, vol. 13 (1906), p. 20.

‡ Burnside, *Quar. Jour. of Mathematics*, vol. 37 (1906), p. 215.

by a number of new theorems relating to the construction of such groups and groups which involve all the substitutions which are commutative with each of the substitutions of a transitive group.*

The investigations of Sylow in regard to groups of a prime degree which were noted in the preceding report have been extended by Frobenius, who proved that there are only four groups of degree p which contain exactly $p+1$ subgroups of order p. This theorem was also proved by de Séguier without the use of group characters in a memoir which, like the one by Frobenius, contains a large number of other new theorems on substitution groups.† From the results of the last two paragraphs it follows that each of the unknown transitive groups of degree p may be assumed both to be multiply transitive and also to involve more than $p+1$ subgroups of order p. Hence these groups contain transitive subgroups of lower degree and therefore belong to a category which has recently been studied by Manning.‡ The special case where $p = 2q+1$, q being a prime, was considered by de Séguier in the memoirs just noted and he arrived at the result that there is no transitive group of degree 23 in addition to the 7 which are well known. This result has been verified by the writer. It may be observed that the smallest possible number of transitive groups of degree $p > 5$ is 6. Jordan proved that this is also the actual number when p is either 47 or 59. Rietz proved that for every value of $p > 11$ of the given form there exists a simple group of composite order which can be represented as a simply transitive primitive group of degree $1 + kp$.§

A fundamental theorem relating to any transitive substitution group G of degree n has been stated as follows: If p^β is the highest power of p which divides n, each Sylow subgroup of order p^α in G has a transitive constitutent of degree p^β and all its other transitive constituents are of degree of $p^{\beta+\gamma}$ ($\gamma \geqq 0$). If $n = 2q^{\alpha_1}$, q being any odd prime, each of the Sylow subgroups whose order is a power of q has just two transitive constituents of degree q^{α_1}. If n is a power of a prime, a Sylow

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* Kuhn, *Amer. Jour. of Mathematics*, vol. 26 (1904), p. 45.

† Frobenius, *Berliner Sitzungsberichte*, 1902, p. 351; de Séguier, *Comptes rendus*, vol. 137 (1901), p. 37, and *Liouville*, vol. 8 (1902), p. 253.

‡ *Transactions Amer. Math. Society*, vol. 7 (1906), p. 497.

§ Rietz, BULLETIN, vol. 11 (1905), p. 544.

subgroup of G whose order is a power of the same prime is transitive.*

The important applications of group characters to such problems as the determination of the least number of variables by means of which a given abstract group may be represented as a linear group, and of all the linear homogeneous substitution groups which are either simply or multiply isomorphic with a given group, have continued to attract many investigators to this field with a view either to extensions of the theory or to making it more accessible. Dickson, Burnside, and Schur have published valuable expository articles on the theory of group characters in which the subject is approached from different standpoints.† While Burnside makes considerable use of Hermitian forms in his fundamental theorems, Schur replaces these by simpler considerations so that he presupposes only an elementary knowledge of matrices from the theory of linear substitutions. Both of these authors employ methods which differ widely from those used by Frobenius, and Schur develops his theory under the heading "New foundation for the theory of group characters."

Frobenius proved various theorems relating to the group characters of the multiply transitive groups which do not involve the alternating group of the same degree and gave a number of historical data relating to these groups. Among these theorems are the following: A character of the symmetric group whose dimension does not exceed $\frac{1}{2}r$ is also a character of every r-fold transitive group of the same degree. The necessary and sufficient condition that a group is either two-fold or four-fold transitive is that it has the character $\alpha - 1$, or the characters $\alpha - 1$, $\frac{1}{2}\alpha(\alpha - 3) + \beta$, $\frac{1}{2}(\alpha - 1)(\alpha - 2) - \beta$ respectively, where $\alpha, \beta, \gamma, \cdots$ are the number of cycles of degrees $1, 2, 3, \cdots$ in a substitution.‡ By means of a fundamental theorem relating to the roots of unity Burnside establishes several theorems which are useful in calculating group characters, and by way of illustration he determines the characters of the simple group of order 504.§ Alasia‖ has also given a brief exposition of the characters of several groups.

* BULLETIN, vol. 9 (1903), p. 543.

† Dickson *Ann. of Math.*, vol. 4 (1902), p. 25; Burnside, *Proc. London Math. Society*, vol. 1 (1903), p. 117; Schur, *Berliner Sitzungsberichte*, 1905, p. 406.

‡ Frobenius, *Berliner Sitzungsberichte*, 1904, p. 558.

§ Burnside. *Proc. London Math. Society*, vol. 1 (1903), p. 112.

‖ Alasia, *Rivista di Fisica, Matematica e Scienze naturali*, anno. 6 (1905), p. 1905.

Loewy published several important memoirs which are principally devoted to the study of the reducibility of the linear homogeneous substitution groups. Most of his theorems apply to groups of infinite order but he has also considered groups of finite order.* One of his theorems was extended by Dickson, who established its validity for much more extensive domains.† All the finite collineation groups in three variables have recently been determined by Blichfeldt, who employs more direct methods than those used by Jordan and Valentiner in their much earlier investigation of this problem. Blichfeldt gives also an enumeration of the principal imprimitive collineation groups in four variables, together with their generating substitutions, and a complete list of the primitive ones based upon various theorems which were developed by him principally in the *Transactions* of this Society.‡

Schur has given a complete solution of the problem of finding all the possible representations of a finite group as a linear fractional substitution group, and, together with Frobenius, he established the following fundamental theorem: A finite group of linear substitutions is equivalent to a real group when its substitutions transform into itself a quadratic form of a non-vanishing determinant, and only then; two isomorphic groups of linear substitutions have the same irreducible components if any two corresponding substitutions have the same trace (Spur) and only then.§ The fundamental theorem proved by Jordan about thirty years ago, which establishes the fact that every finite linear homogeneous group on n variables contains an invariant abelian subgroup such that the order λ of the corresponding quotient group is less than a certain number depending only upon n, has been extended both by Blichfeldt and by Schur. The former has found further restrictions for λ in the article published in the *Transactions* to which we have just referred, as well as in an earlier article in the same journal, while the latter has considered the possible orders of the linear groups when the trace of each substitution is restricted to certain fields.

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* Loewy, *Transactions Amer. Math. Society*, vol. 4 (1903), pp. 44 and 171; ibid., vol. 6 (1906), p. 504; Verhandlungen des dritten internationalen Mathematiker-Kongresses (1904), p. 194.

† Dickson, *Transactions Amer. Math. Society*, vol. 4 (1903), p. 434.

‡ Blichfeldt, *Mathematische Annalen*, vol. 63 (1907), p. 552; ibid., vol. 60 (1905), p. 204; *Transactions Amer. Math. Society*, vol. 6 (1905), p. 230.

§ Frobenius and Schur, *Berliner Sitzungsberichte* (1906), p. 186; *Crelle*, vol. 127 (1904), p. 10.

In this case it is possible to find a number which is divisible by every possible group and which depends only upon the field and the number of variables.*

Shaw has investigated the algebras defined by finite groups. "By a group algebra is meant that linear algebra whose units are defined to be such that each unit e_i corresponds to an operator O_i of some given finite abstract group, and conversely, and such that to each equation of the group $O_iO_j = O_k$ corresponds an equation $e_ie_j = e_k$ of the algebra." In this investigation Shaw confines himself to the scalar continuous field and observes that if the coefficients are any numbers in such a field, every abelian group of the same order gives the same group algebra, a result which is not true for all fields of coefficients. He points out the ultimate connection between group algebras and the theory of group characters and group determinants, and cites Poincaré's fundamental theorems bearing on such algebras.†

In volume 23 of the *Mathematische und Naturwissenschaftliche Berichte aus Ungarn*, Visnya determines a necessary and sufficient condition that a finite group of linear substitutions is intransitive and he also considers all the possible Hermitian invariants of such a group. A sufficient condition for its intransitivity is due to Maschke and was published much earlier. Burnside has recently given some new criteria for the finiteness of the order of a group of linear substitutions. The coefficients in the substitutions of a group of homogeneous linear substitutions are generally complex numbers of the form $\alpha + \beta i$, where α and β are real numbers. If the group is of finite order, there is a finite number of coefficients and therefore there is a finite positive number M such that for each coefficient $|\alpha| < M$ and $|\beta| < M$. Similarly there must be another positive number m such that for each coefficient $|\alpha| = 0$ or $> m$ and $|\beta| = 0$ or $> m$. The existence of these two numbers is proved to be both a necessary and a sufficient condition that the group is finite. Another necessary and sufficient condition is expressed by the following theorem: "If, in a group of linear substitutions on a finite number n of symbols, the order of every substitution is equal to or less than a finite number m, then the group is of finite order." Burnside

*Schur, *Berliner Sitzungsberichte* (1905), p. 1.
†Shaw, *Transactions Amer. Math. Society*, vol. 5 (1904), p. 326.

proves this theorem * without giving reference to the less general theorem proved by Loewy which is mentioned at the end of my report on the groups of an infinite order.† Another test given in this paper for the finiteness of a linear substitution group on a finite number of symbols is that it contains a finite number of distinct sets of conjugate substitutions.

Quite recently Loewy has investigated the groups of linear homogeneous substitutions which are of the type of a finite group and gave a complete development of their theory.‡ Dickson has recently published two papers in which he considers for the first time the problem of representing a given finite group as a linear congruence group.§ He points out that the only one of the different expositions of Frobenius's theory of group characters mentioned above which may be utilized in the construction of a corresponding modular theory is that by Schur. While the developments of Frobenius relate to the representation of a given finite group as a non-modular linear group the work of Dickson employs a modulus in such a representation.

UNIVERSITY OF ILLINOIS,
July, 1907.

THE DRESDEN MEETING OF THE DEUTSCHE MATHEMATIKER-VEREINIGUNG.

THE 1907 meeting of the Deutsche Mathematiker-Vereinigung was held in Dresden, September 15–21, in conjunction with the 79th convention of the Naturforscher und Aerzte. The meeting took place in room 80 of the Technische Hochschule. In commemoration of the 200th anniversary of Euler's birth a considerable number of the papers were devoted to an exposition of his services to science. The following papers were read:

1. K. ROHN, Leipzig: "Algebraic space curves" (report).
2. F. KLEIN, Göttingen: "Concerning the connection between the so-called theorem of oscillation of differential equations and the fundamental theorem of automorphic functions."

* Burnside, *Proc. London Math. Society*, vol. 3 (1905), p. 435.
† BULLETIN, vol. 7 (1900), p. 121.
‡ Loewy, *Mathematische Annalen*, vol. 64 (1907), p. 264.
§ Dickson, *Transactions Amer. Math. Society*, vol. 8 (1907), p. 389.

3. G. LANDSBERG, Kiel: "Theory of curvature and calculus of variations."

4. A. v. BRILL, Tübingen: "Introduction to the Euler commemoration."

5. L. SCHLESINGER, Klausenburg: "On a problem of diophantine analysis, as considered by Fermat, Euler, Jacobi, and Poincaré."

6. A. PRINGSHEIM, Munich: "Concerning Euler's transformation of series."

7. E. BRAUER, Karlsruhe: "Euler's theory of the turbine."

8. F. S. ARCHENHOLD, Treptow: "Some letters of Euler."

9. R. GANS, Tübingen: "Euler as physicist."

10. H. E. TIMERDING, Strassburg: "Euler's investigations in nautical mechanics."

11. W. HORT, Gross Lichterfelde: "Euler's significance in technical science."

12. E. HOPPE, Hamburg: "Euler's place in the theory of optics."

13. L. SCHLESINGER, Klausenburg: "Development of the analytic theory of linear differential equations since 1863" (report).

14. A. SCHOENFLIES, Königsberg: "On the so-called Richard paradox in the theory of point sets."

15. F. HAUSDORFF, Leipzig: "On dense types of order."

16. H. WIENER, Darmstadt: "Geometric theory of invariants of binary forms."

17. V. Varicak, Ogram: "Contributions to non-euclidean geometry."

1. Professor Rohn's report will soon appear in the Encyklopädie der mathematischen Wissenschaften, III, C. 8.

4. In a short paper Professor v. Brill called attention to Euler's birthplace, briefly sketched his life, expressed the belief that Germany's celebration in Euler's honor would find comprehending appreciation throughout the civilized world, and discussed its significance for the history of science.

7. Professor Brauer's paper will appear in the *Jahresbericht*.

9. Professor Gans gave the following historical summary: Euler published important memoirs in nearly all fields of physics. Above all, he created analytic mechanics, which, in

contrast with the earlier geometric method, supplies an ever-ready recipe for new as well as old problems. Before Euler's time even geniuses contented themselves with easy problems; since then, anyone can calculate by rule the path of a given mass-point under the action of determinate forces. Applying his theory to astronomy, Euler calculated by the method of variation of the constants, the perturbative effect of celestial bodies upon the orbit of a planet. He rendered assistance to the science of artillery by his approximative treatment of the path of a projectile; to nautical science by his theory of the ebb and flow of tides and of the building and sailing of ships. He investigated the bending and elastic swing of beams and calculated the safety load of a column, thus rendering service to technical science. He studied the theory of sound and investigated the physical basis of musical consonance and dissonance. The results of his work on optical instruments, which fill three thick volumes, have contributed much to the technique of microscope and telescope making, and, above all, have resolved the physics of catoptrics to that of dioptrics. His discovery of the achromatic telescope deserves especial mention, since it was at Euler's instigation that Dollond made his glass. Euler's ether theory is of great importance for theoretical physics and natural philosophy. He assumes the universe filled with a fine substance to whose vibrations the phenomena of light are due and whose currents are the ultimate cause of gravitation, electricity, and magnetism. It is not to be wondered at that this ether theory seems imperfect from our present standpoint, for in Euler's time Coulomb's law and the connection between light, electricity, and magnetism were unknown. Besides physical proofs Euler sometimes used theologic-teleological proofs, although his mathematical-analytic interest is frequently in the foreground. But in cases of application to astronomy or technical science no method is too uninteresting for him. We admire in him the powerful physical analyst, who studied the details, and the great natural philosopher, who sought the unities of natural phenomena.

10. Professor Timerding discussed those works of Euler which treat of naval mechanics. He indicated the principal points of view from which these works, especially the Scientia navalis, 1749, and the Théorie complète de la construction et de la manoeuvre des vaisseaux 1773, may be judged. Two things must be con-

sidered — personal relation to Euler's scientific temper and to his other investigations, and technical relation to other investigations concerning naval mechanics and to the development of rational naval construction. One should emphasize, for example, the relation to Euler's discovery of the laws of motion of rigid bodies and, on the other hand, the impulse to a critical investigation of the strain of the ship by reason of the coming into play of different forces. It should also be mentioned that Euler knew all three kinds of practicable propellers and had calculated their efficiency. The author emphasized the significance of Euler's systematic development for the subsequent progress of the theory of the ship.

No. 11 is to be published at once in the *Physikalische Zeitschrift*.

12. Professor Hoppe drew attention to the fact that Euler was the first to treat the vibrations of light analytically and to deduce the equation of the curve of vibration as dependent upon elasticity and density. Euler distinguished simple rays of homogeneous wave length from those of white light, calling them elementary rays generating elementary colors. He deduced the law of refraction analytically and explained that the rays of greater wave length must suffer the least deviation. He also investigated non-transparent bodies with regard to their behavior toward light. He studied dispersion in the search for a corrective for chromatic aberration, which Newton had declared unattainable. For this purpose he found the formula that the ratio of the natural logarithms of the indices of refraction must remain constant for different colors. If the ratio is α then the dispersion is given by $dn = (1-\alpha)(1+\alpha)^{-1} n l n$. It was this investigation that induced Dollond to construct his achromatic lenses. Euler calculated a large number of composite telescopes and microscopes, thereby increasing the possibilities in the construction of these instruments. The results of his investigation with the composite ocular are of special importance. Euler treated atmospheric refraction of light analytically both for celestial and terrestrial objects, finding for the latter an expression that is fundamental in leveling. With regard to photometry Euler distinguished between intensity of light and of illumination and discovered the dependence of the latter upon the inclination of the surface and the decrease of the

former with the distance. He also called attention to the optical methods for eliminating the third dimension. Euler was thus the only physicist of the eighteenth century who advanced the undulatory theory.

14. Professor Schoenflies exposed the fallacy in the reasoning of Richard and showed that a parallel line of argument could be made to invalidate the results of all human thought.

15. Denoting by *series* (Reihen) sets of the type of Cantor's "Zahlclassen," or of the inverse type, one can assign a definite character to each element and to each gap of a dense set, according to the series which the element, or the gap, bounds to right or left. Upon this Professor Hausdorff based a classification of dense sets and an existence proof for the a priori obvious kinds. In the first case, in which the dense set contains series of potency one, there are two different species, represented by the linear continuum and by the set of rational numbers. In the case of series of potency two, there exist 210; of potency three, 243376 different species.

17. Professor Varicak's paper is in abstract as follows. Erect two normals to the x-axis at points $N_1(n_1, 0)$, $N_2(n_2, 0)$. There will be four straight lines gi parallel to these normals, uniquely determined by the intercepts n_1, n_2 and by the sense in which the gi shall be parallel. Making use of Lobachevsky's coordinates and assuming $\epsilon_1(\epsilon_2) = +1$ when the first (second) end of gi lies in the positive half plane, but equal to -1 in the contrary case, we can write the equation of these four parallels as follows:

$$e^x + \epsilon_1\epsilon_2 e^{n_1+n_2}e^{-x} - (\epsilon_1 e^{n_1} + \epsilon_2 e^{n_2})\,\mathrm{sh}\, y = 0.$$

This is the general equation of the straight line which can often be used to advantage. In the second part of his paper the author characterized some movements of the Lobachevsky plane by means of infinitesimal transformations. Translations along the "Abstandslinie," the normal and the parallel to the x-axis respectively are given by

$$Uf \equiv \frac{1}{\mathrm{ch}\, b}\cdot\frac{\partial f}{\partial x}, \quad Uf \equiv \frac{\partial f}{\partial y}, \quad Uf \equiv (1 - e^{-2x}\mathrm{sh}^2 b)\frac{\partial f}{\partial x} - \mathrm{sh}\, y\frac{\partial f}{\partial y}.$$

Rotations about the positive end of the x-axis and about the origin are given respectively by

$$Uf \equiv \frac{\sqrt{e^{2(x-b)}-1}}{e^{2(x-b)}} \cdot \frac{\partial f}{\partial x} + \frac{1}{\operatorname{ch} y} \cdot \frac{\partial f}{\partial y},$$

$$Uf \equiv -\frac{\operatorname{sh} y \operatorname{ch} x}{\operatorname{sh} p} \cdot \frac{\partial f}{\partial x} + \frac{\operatorname{sh} x}{\operatorname{sh} p} \cdot \frac{\partial f}{\partial y}.$$

Invariant pencils of curves can be determined easily by Lie's method.

At the business meeting of the Vereinigung a committee, consisting of Professors Pringsheim, Stäckel, and Krazer, was appointed to consider ways and means looking toward the publication of Euler's works. It was hoped that a report of progress could be made at the meeting in Rome next April. Professor Klein was elected to succeed Professor Pringsheim in the Vorstand and will be chairman for the current year. Professor Krazer was continued as Secretary.

During their stay in Dresden the visiting scientists did not lack for entertainment. The evenings were socially filled as follows: Sunday, September 15th, there was an informal initial gathering in the main hall of the permanent exposition building for the purpose of renewing old acquaintanceships and forming new ones. On Monday evening the Belvedere was elaborately decorated with colored lights in honor of the visitors. By reason of inclement weather the open air concert had to be abandoned, however, in favor of indoor entertainment. Tuesday evening the Naturforscher were the guests of the King of Saxony at a superb presentation of La Bohême at the royal opera house. Wednesday evening the visitors banqueted themselves, Thursday evening they were received at supper by the mayor of Dresden, in the main hall of the exposition building. During the entire week the city of Dresden, with its wealth of attractions, was hospitably open to the visitors, while abundant opportunity was offered to those who desired to make excursions, at reduced rates, to neighboring points of interest.

C. A. NOBLE.

GENERAL THERMODYNAMICS.

Thermodynamics: An Introductory Treatise Dealing Mainly with First Principles and their Direct Applications. By G. H. BRYAN. Teubner's Sammlung, Band XXI. Leipzig, 1907. xiv + 204 pp.

The books on thermodynamics are legion. In purpose and content they run the whole gamut from Maxwell's inimitable little work on heat and the many no more advanced though greatly inferior treatments of the subject to the large treatises like Weinstein's Thermodynamik und Kinetik der Körper in three sizable and highly mathematical volumes of which the last is not yet finished. Some are chiefly concerned with engineering problems, others with physics, and lately many with physical chemistry. One might therefore well ask what was left for a new work on thermodynamics except repetition. Anybody who has undertaken to teach a comprehensive course on thermodynamics, beginning essentially at the foundations and covering as well as may be the various applications up to and even including those in physical chemistry, will have no hesitation in answering that there is no short treatise which he could use as a guide — as a précis — for his whole course, nothing that at once was comprehensive and compact, as deductive as might be and yet thoroughly physical, and furthermore well balanced between the distracting needs of the different dependent fields of science. Better might it be said that such would have been the answer a few months ago. Now it would no longer be true; for Bryan's little book furnishes precisely this desideratum.

It should be noted that there is no necessity of banishing the physical side of a science in giving a deductive presentation of its theory. The history of most branches of mathematical physics is about this: First some fundamental experiments combined with a very crude theory which is frequently almost as much wrong as right; then after an increasing accumulation of facts there appears some master mind to formulate a comprehensive and essentially correct theory based upon a few crucial experiments; finally the deductive stage arrives in which the start is made from certain mathematical statements and physical experiment no longer plays any considerable rôle. This

last stage is, however, that in which physical *interpretation* should be most in evidence, and it is the omission of this constant a posteriori contact with nature that renders so much mathematical physics futile as far as physical science is concerned. The advance must not be away from physics and into mathematics; it should be from a priori physics and a posteriori mathematics to a priori mathematics and a posteriori physics. This last point of view is apparently that adopted by Bryan in regard to thermodynamics, as it is by most modern authors in regard to mechanics and the classical theories of optics.

The author divides his work into three parts: Physical aspects of thermodynamics, pages 1–26. The foundations of rational thermodynamics, pages 27–96. Thermodynamics of particular systems, pages 97–198.

From the title of the first part it might appear that we were still in the stage of a priori physics and close contact with experiment. A perusal of the text will rapidly disabuse one of any such notion. As a matter of fact only such reference to material bodies is made as is necessary for the formulation of the definitions and laws upon which thermodynamics is to be built. It would not do to say that in this first part the author lays down axioms for a thermodynamics; his procedure, subject to later revision, is analogous to Newton's in mechanics. Heat is taken as a fundamental idea just as force was. In fact as we find in mechanics the law that, when force acts, the amount of momentum is changed by the amount of the impulse of the force, so we find here the law that, when heat is transformed into work or conversely, the quantity of heat gained or lost is proportional to the quantity of work lost or gained. And there is the immediate additional statement that this law affords a measure of heat just as Newton's law gives a measure of force. Some might wish to banish heat from thermodynamics as they have banished force from dynamics. The attempt to do so does not seem particularly valuable from a pedagogic point of view, and it is probably well that Bryan is content at this early stage of his work to stay his desire for deduction just where he does. The concluding chapter of part I deals with the necessary changes of variable incident to the transformation of the many thermal coefficients previously defined.

Part II begins with an admirable discussion of various behaviors of energy — conservation, localization, reversibility,

availability, and the dependence of the last on external conditions.* This might all find its way quite appropriately into many treatises on mechanics. The author then proceeds to discuss irreversibility with the definition that a change is said to be irreversible when the system cannot of itself change back to its original condition, and he lays down two axioms (from now on the word axiom is often in evidence): 1) If a system can undergo an irreversible change it will do so. 2) A perfectly reversible change cannot take place of itself; such a change can only be regarded as the limiting form of an irreversible change. It seems as though the second axiom more or less vitiates the immediately preceding definition of irreversibility. Although the difficulty is obviated before many pages have been turned, it might have been better to phrase the statements a little more carefully. The same may be said of the later sentence which expresses the fundamental principle of degradation: In all irreversible transformations available energy is lost, never gained. If anything in the way of emphasis is to be added to lost, it would be well to say something as definite as never gained or conserved. Lost and gained do not together form a universe of discourse in this case. These, however, are small matters to throw in the face of an author who is perhaps the first to have thoroughly conscientious scruples as regards definitions and axioms in his upbuilding of thermodynamics.†

* The author's characterization of energy on page 39 is far from possessing the completeness and definiteness which is desirable in an entity that is to be made the basis of his whole deductive system. To critics inclined to be too severe at this point, may be replied with great reason that in the present state of our knowledge it is apparently impracticable to build up a perfectly definite system of rational thermodynamics. Indefiniteness is bound to occur somewhere unless the number of undefined elements is made larger than Bryan wishes to make it. To leave energy a somewhat vague and elastic conception is probably far better than to have some other entity obscure or undefined — and for precisely the reason that energy is so familiar and fundamental in all physics as to spare definition better than anything else. Furthermore, from a logical point of view, it should be noted that with the statements on page 39 must be included all the later axioms which go to contribute additional precision to the basic undefined symbol "energy."

† It is only fair to mention at this point Perrin's Traité de chimique physique: les principes, Paris, Gauthier-Villars, 1903. Perrin is here writing for chemists. He goes so carefully into all the fundamental ideas of thermodynamics as to make his book almost tedious reading for anybody who knows as much about energy and entropy as a chemist ought to; but as the chemists do not know as much about such things as they should, the author is doubtless forced to follow the laborious detail that he does. Furthermore Perrin chooses to keep in much closer contact with experiment than Bryan and to

After some further treatment of the characteristics of a system we call thermodynamic, there appears the definition of the quantity of heat involved in a thermodynamic transformation, namely,

$$dQ = dU + dW,$$

where dU is the increase of energy in the system and dW the amount of external work done by the system. A discussion of the Carnot's cycle in its relation to the principle of the degradation of energy, which the author adopts as more fundamental than the usual statement of the second law, then leads to the definition of temperature and of absolute temperature (thermodynamic scale) in particular. The author's presentation of this whole introductory matter is certainly not hackneyed. It can only be characterized as original. It is extremely illuminating and peculiarly satisfactory to the mathematician who naturally demands closer formal reasoning than the physicist. It is to be hoped that future treatments of the subject will borrow much of the clearness and logic of Bryan's even in such frequent cases as those in which a complete adoption of his method would be inadvisable for one reason or another.

Thus far we have covered six chapters. The remaining chapters of this part may be spoken of more briefly. They deal with the usual topics. Chapter VII takes up the relation of available energy to entropy. Two definitions of entropy are given and each is elaborated. Homogeneous and heterogeneous systems are alike treated in detail, and so with both reversible and irreversible transformations. This chapter is long and clear; every reader has ample opportunity to become so thoroughly familiar with entropy as to forget that it ever was considered a weird bugaboo. The following two chapters deal with the matters of stability and thermodynamic potentials.

Part III is on the thermodynamics of particular systems. That the author should start off with the treatment of radiation is a bit of brave novelty. As a matter of fact such systems

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depend less on rational dynamics; his work is not so completely deductive. This again is probably the wisest course for him to follow. To any who find difficulty in Bryan's Thermodynamics, especially in these underlying chapters, Perrin's Principes is heartily to be recommended; and even for those who find no trouble in following Bryan, Perrin has much of interest to offer.

are perhaps the simplest, and the deduction of Stefan's law with the further proof that optical methods cannot increase availability gives to the subject at the very start an interest and vigor which the older treatments of perfect gases cannot approach. The succeeding chapter is largely a collection of formulas expressing the various thermal coefficients of simple systems in terms of the potentials. Then follows in chapter XII some account of gaseous systems whether pure or mixed. No very great detail is entered upon; true to the author's general intention the chapter is merely a sketch. Chapter XIII contains an account of the determination of thermodynamical quantities and will be especially welcome to those who like to see how the fundamental magnitudes are derivable from experimental data.

The next three chapters, XIV, XV, XVI, touch on the physico-chemical aspect of thermodynamics. The first takes up van der Waals's equation and the critical states, and also the phase rule and triple point for a single substance. The second treats the general question of phase equilibrium for heterogeneous substances. The third, which deals with reversible thermoelectric effects, gives Helmholtz's and Gibbs's formulas for reversible elements and an excellent discussion of the Kelvin and Peltier effects. In the next and last chapter are found some geometric and dynamical representations of thermodynamic phenomena. As might be expected these are for the most part Gibbs's surfaces and related diagrams on the one hand and the cyclic dynamical systems on the other. It seems as if it might have been well to add just a few words more on strictly statistical methods with perhaps a mention of the analogies to thermodynamics which Gibbs pointed out in his last work.* It should be borne in mind, however, that to say merely a few words which shall be comprehensible is a very difficult matter in a subject so technical and formulaic as statistical mechanics.

There remains merely a short conclusion, which again and for the last time emphasizes the fundamental points of the text. An elaborate index closes the volume. Although this is scarcely necessary in a work so concise and so well arranged as the present, it is surely not in the way.

From the foregoing sketch of the contents and general aim of Bryan's two hundred pages, it may be possible to gain some

* Elementary Principles in Statistical Mechanics, Charles Scribner's Sons, New York, 1907.

insight into the character of the work, and to see that an empty place in our scientific literature has been discovered and satisfactorily filled. As was stated before, this book furnishes an excellent précis for anybody lecturing on general thermodynamics. If in addition there were in existence an elaborate treatise of three or four times the length following the same order of arrangement and amply supplied with physical applications and developments, the student of thermodynamics would scarcely have need of any lectures. As it is, both student and teacher are so much better off than they were a twelvemonth ago that it would be a bit ungracious already to ask for more — or would it be the highest expression of gratitude?

EDWIN BIDWELL WILSON.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
BOSTON, *September*, 1907.

SHORTER NOTICES.

Funzioni poliedriche e modulari. By G. VIVANTI. Milan, Hoepli, 1906. viii + 437 pp.

THE object of this work cannot better be characterized than by the following quotation from the preface:

"The present volume has the modest scope of preparing the reader for the study of the classic lectures of Klein on the icosahedron and of the treatise on the elliptic modular functions by Klein and Fricke. The first of these, a model of elegance and a veritable mine of new and interesting ideas, is quite difficult to read both on account of the many concepts which are barely outlined and still more because, even after the various details have been understood, the connection which binds them, the guiding idea, is far from evident and comes to light only after a profound study and a thorough review of the entire material. The second work, on account of the mass and the multiformity of its contents, does not lend itself readily to an introductory study. On account of the lack of a suitable treatise for facilitating acquaintance with the theory of the polyhedral and modular functions, I believed it my duty to make available for others the work of elaboration which I have performed for my own use, thus saving them the repetition of this useful but laborious work.

"It may be too bold an attempt to include in the brief space of a small manual the principles of two vast and important theories. Nevertheless it seemed useful, in spite of the greater conciseness required, to unite both theories in order to avoid the repetition necessary in treating them separately, and also to show that the icosahedron theory, which is capable of appearing in the field of analysis as an elegant and independent creation of a special type, is only the first of a series of related constructions very closely bound together. From this point of view it would have been better still to take another step and include also the theory of the automorphic functions; but this would clearly be impossible.

"To save space, and for greater symmetry of treatment, I have assumed that the reader, besides having a certain familiarity with the more elementary parts of mathematics, has also some notions of several more advanced theories: analysis situs, functions of a complex variable and Riemann surfaces, elliptic functions, abelian integrals, linear differential equations, theory of numbers."

The program which the author thus places before himself has been carried out, we think, with much care and good judgment. He has treated his subject with admirable simplicity, directness, and unity. Not only will this work greatly facilitate the efforts of the reader to master these comprehensive theories, but it will be of particular service to those whose main interests lie outside this special field, by enabling them to become familiar with its general topography without an undue expenditure of time.

The book is printed in large, well leaded type. By using a paper as thin as is consistent with opaqueness and by cutting down to a narrow margin, the publisher has produced a small and handy volume without any sacrifice of legibility.

J. I. HUTCHINSON.

Vorlesungen über Zahlentheorie. Einführung in die Theorie der algebraischen Zahlkörper. By J. SOMMER. Leipzig, B. G. Teubner, 1907. iv + 361 pp.

THE generalizations of the ordinary theory of numbers which, following Gauss's introduction of complex integers, have been made by Kummer, Dirichlet, Dedekind, and Kronecker constitute an extensive and exceedingly interesting part of mathe-

matics.* Nevertheless it is true that this subject is not as widely read as is proper in view of its importance; and undoubtedly the reason is to be found in its abstract nature.

The recent books of Bachmann † and König ‡ upon this general subject are excellent, but many will find them very difficult because of the generality of their treatments. Indeed, Professor Dickson in reviewing them § felt it necessary to give a brief exposition of the theory for a special case, and there is no doubt of the desirability of an extensive treatment of the simpler cases which will lead up to the general discussion. The volume under review has been prepared to serve those who have hitherto been hampered by the lack of an introduction to the subject.

The author has successfully undertaken to present the chief points of the theory by means of the elementary cases of (absolute) quadratic and cubic number fields and fields quadratic with respect to a quadratic fundamental field. The book is divided into five chapters, the first of which constitutes an exceedingly brief introduction, so brief in fact that it must be regarded as simply a statement of those topics of the ordinary theory of numbers whose generalizations are to be given later and are to form the foundation of the discussion. And, naturally enough, these topics are the divisibility of integers, the function $\phi(n)$, congruences, and Fermat's theorem.

The second and third chapters constitute two thirds of the book and are the most important ones. In the second the theory of quadratic number fields is worked out in detail in connection with the topics above mentioned, and in this chapter the reader is given an opportunity to become intimately acquainted with the ideas of number field, ideals, and the generalizations of the results obtained by Fermat and Gauss. In chapter three applications of the theory of quadratic fields are made to several important problems, and particularly welcome to many will be the introduction of the famous "last theorem" of Fermat.

To give the reader a closer approach to the generalities of the subject, the author devotes a fourth chapter to such a broadening of methods as will make the treatment adequate for the

* See Hilbert, *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 4.
† Zahlentheorie, Teil 5, 1905.
‡ Einleitung in die allgemeine Theorie der algebraischen Grösen, 1903.
§ BULLETIN, April, 1907.

case of cubic number fields. A fifth chapter deals with fields which are quadratic with respect to another one, itself a quadratic field. Both of these chapters are relatively short and are not intended to be exhaustive.

The value of the volume is distinctly increased by the introduction of a considerable number of simple illustrations of the theory in addition to the more elaborate applications made in the third chapter.

The book is timely, has been put out by the publishers in an attractive form, and should materially increase the number of those who will undertake to familiarize themselves with this branch of general algebraic theory.

GEORGE H. LING.

Les Principes des Mathématiques, avec un appendice sur la philosophie des mathématiques de Kant. Par LOUIS COUTURAT. Paris, Felix Alcan, 1905. viii + 310 pp.

THE present work owes its origin, according to the author, to the appearance of "the masterly work of Bertrand Russell which bears the same title." It seems that originally it was intended to be merely a review of the older work. But it is much more than a review of Russell's treatise. We can readily sympathize with the author when he tells us in the preface that in commenting and illustrating Russell's theories he was led gradually to include in his review abstracts of most of the recent papers dealing with the same questions. The result has been that the author has written a comprehensive and careful report on the present state of the logical foundations of mathematics, which on account of its clear style and admirable arrangement of content is valuable, not merely as a work of reference but also as a book well adapted to the needs of anyone desiring to acquaint himself with the fundamental ideas and methods of the subject with which it deals.

The book is divided into six chapters, to which are added two "Notes" and an appendix. The first chapter deals with the principles of formal logic, as developed by Peano and others. The next four chapters discuss respectively the notions of number, order, the continuum, and magnitude. The last and by far the longest chapter deals with the foundations of geometry. It is divided into four parts treating respectively of the dimensions and topology, projective geometry, descriptive geometry,

and metric geometry. This should give an idea of the scope of the book. Dealing as it does with an order of ideas that is familiar to the readers of these pages, a more detailed analysis of the contents would be out of place. We regret the absence in the book of an alphabetical subject index.

J. W. YOUNG.

NOTES.

THE Annual Register of the AMERICAN MATHEMATICAL SOCIETY is now in preparation and will be issued in January. Blanks for furnishing necessary information have been sent to the members. Early notice of any changes since the issue of the last Register will greatly facilitate the work of the Secretary.

THE Chicago Section of the AMERICAN MATHEMATICAL SOCIETY will hold two joint sessions with Sections A and D of the American association for the advancement of science on Monday and Tuesday, December 30–31, 1907, at the University of Chicago. The speakers on Monday afternoon will consider the present status of the teaching of mathematics to students of engineering, both in this country and abroad. On Tuesday morning there will be a symposium on the topic: "What is needed in the teaching of mathematics to students of engineering?" The topics included will be (*a*) What branches of mathematics are desirable? (*b*) To what extent should the various branches be taught? (*c*) What methods of presentation are needed? (*d*) What are the chief ends to be attained?

Prominent practical engineers and teachers of mathematics and engineering will take part in the programme. There will be a banquet to engineers and mathematicians on Monday evening. Further details will be announced in the printed program, which will be distributed well in advance of the meeting.

TITLES and abstracts of papers intended for presentation before Section A of the American association for the advancement of science at the coming Chicago meeting should be sent to Professor G. A. MILLER, 907 West Nevada street, Urbana, Ill., who has been appointed secretary of Section A to succeed Professor L. G. WELD, who has retired after several years' service in that office.

THE second regular meeting of the Southwestern Section of the AMERICAN MATHEMATICAL SOCIETY will be held at Washington University, St. Louis, November 30, 1907. A meeting of the Central association of teachers of science and mathematics will be held on Friday and Saturday of the same week at the McKinley high school. The programmes have been adjusted for the convenience of those who desire to attend both meetings.

THE concluding (October) number of volume 8 of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY contains the following papers: "The expansion of a function in terms of normal functions," by MAX MASON; "Sur les opérations linéaires (troisième note)," by MAURICE FRÉCHET; "The elliptic integral in electromagnetic theory," by A. G. GREENHILL; "Notes and errata, volume 8."

Professor MAXIME BÔCHER has been elected editor in chief of the *Transactions*, succeeding Professor E. H. MOORE, who has held this office since the journal was founded in 1899.

THE opening (October) number of volume 9 of the *Annals of Mathematics* contains the following papers: "The calculation of logarithms," by J. K. WHITTEMORE; "Cubic curves in reciprocal triangular situation," by J. F. MESSICK; "The groups generated by two operators such that each is transformed into its inverse by the square of the other," by G. A. MILLER.

AT Parma, September 23–28, the first regular meeting was held and the organization perfected of the Italian association for the advancement of science (Società italiana per il progresso delle scienze). It is composed of fourteen sections, the first including mathematics, astronomy and geodesy. The first president of the association is Professor V. VOLTERRA, of the University of Rome, with Professor M. CERRUTI, of the University of Rome, as president of section A. The following mathematical papers were read at this meeting: By U. AMALDI, "Lie's theory of continuous groups of transformations" (report); by E. BORTOLOTTI, "Plans for the publication of the works of Ruffini"; by P. BURGATTI, "Some points in the theory of differential equations"; by G. FUBINI, "Recent methods for the solution of Dirichlet's problem"; by A. GARBASSO, "Mirage and the optics of heterogeneous media"

(report); by G. LAURICELLA, "On functional equations"; by T. LEVI-CIVITA, "On electromagnetic mass" (report); by R. MARCOLONGO, "Mathematical theory of elasticity"; by C. SOMIGLIANA, "On the mathematical preparation of students of engineering"; by O. TEDONE, "Differential equations of mathematical physics"; by G. VAILATI, "Mathematical instruction in the middle schools."

THE educational museum of Teacher's College, Columbia, has prepared a second list of lantern slides illustrating the development of mathematics. Of the present series, Nos. 120–145 are pictures of sample pages of old text-books, etc., Nos. 145–207, old mathematical instruments, Nos. 208–232, modern mechanical calculation, Nos. 253–265, the development of analytics and the calculus. The 160 plates of the present series can be obtained on the same terms as those of the earlier one (see BULLETIN, vol. 13, page 410).

AMONG the books just issued in Professor Cattell's Library of philosophy (Scribner, New York) are translations by Professor G. B. HALSTED of H. Poincaré's two recent monographs: Science and hypothesis (196 pages) and The value of science (147 pages).

THE following books are announced to appear this fall: M. BÔCHER, Higher algebra, Macmillan; E. J. TOWNSEND, Differential and integral calculus, Henry Holt; W. J. BERRY, Differential equations of the first species, Van Nostrand.

UNIVERSITY OF PARIS. *First semester*, beginning November 4. — By Professor G. DARBOUX: Deformation of surfaces, two hours. — By Professor E. GOURSAT: Elements of the theory of analytic functions, two hours. — By Professor P. PAINLEVÉ: General laws of motion and equilibrium, two hours. — By Professor P. APPELL and Dr. E. BLUTEL: General course in mathematics, one hour. — By Professor L. RAFFY: History and methods of analytic geometry, two hours. — By Professor H. POINCARÉ: Theory of the moon, two hours. — By Professor J. BOUSSINESQ: Analytic theory of heat, two hours. — By Professor G. KOENIGS: Thermodynamic theory of machines, two hours. — By Professor E. BOREL: Theory of probabilities, two hours. — By Professor J. TANNERY: Differential and integral calculus, two hours. — By Professor J.

HADAMARD: General mathematics, two hours. Conferences, two hours per week, will be held by Professors Raffy, Hadamard, Puiseux and Drs. Blutel, Servant. *Second semester*, beginning March 10, 1908. — By Professor E. PICARD: Principal types of differential equations of mathematical physics, two hours. — By Professor E. GOURSAT: Partial differential equations, two hours. — By Professor P. PAINLEVÉ: Analytic mechanics, two hours. — By Professor P. APPELL: Analysis and mechanics, two hours. — By Professor L. RAFFY: Differential properties of space curves, two hours. — By Professor P. ANDOYER: Mathematical astronomy, two hours. — By Professor J. BOUSSINESQ: Theory of periodic vibrations, two hours. — By Professor G. KOENIGS: Kinematics, two hours. — By Professor J. TANNERY: Differential and integral calculus, two hours. — By Professor E. BOREL: Mechanics, two hours. — By Professor J. HADAMARD: General mathematics, two hours. The last three courses will be given in the Ecole normal both semesters. Mathematical conferences same as first semester.

PROFESSOR G. MORERA, of the University of Turin, has been elected a member of the royal academy dei Lincei at Rome.

PROFESSOR E. PASCAL, of the University of Pavia, has been appointed professor of higher analysis at the University of Naples.

PROFESSOR G. TORELLI, of the University of Palermo, has been appointed professor of the calculus at the University of Naples.

DR. G. VITALI has been appointed docent in the calculus at the University of Genoa.

AT the recent annual meeting of the Cambridge philosophical society, Dr. E. W. HOBSON was reëlected president and Dr. E. W. BARNES secretary.

PROFESSOR H. M. TORY, of McGill University, has resigned to accept the presidency of the new provincial university of Alberta, Canada.

DR. R. C. MACLAURIN, of the University of Wellington, New Zealand, has been appointed professor of mathematical physics at Columbia University.

DR. J. A. EIESLAND, of the U. S. Naval Academy, has been appointed professor of mathematics at the University of West Virginia.

DR. S. G. BARTON, of the University of Pennsylvania, has been appointed to the chair of mathematics in the Clarkson School of Technology, Potsdam, N. Y.

DR. J. F. MESSICK, late of Williams College, has been appointed professor of mathematics at Randolph Macon College, Va.

DR. W. A. MANNING, of Stanford University, has been promoted to an assistant professorship of mathematics.

AT the University of Maine, Mr. H. R. WILLARD has been promoted to an assistant professorship of mathematics.

AT Yale University, Mr. E. W. SHELDON, Mr. H. T. BURGESS and Mr. J. C. RAYWORTH have been appointed instructors in mathematics.

AT the University of California, Professors G. C. EDWARDS, C. A. NOBLE, and A. W. WHITNEY have been granted leave of absence for one year. Mr. A. J. CHAMPREUX has been promoted to an instructorship. Messrs. J. LIPKE, J. H. TENER, and B. A. BERNSTEIN have been appointed instructors, and Messrs. E. F. A. CAREY and H. W. STAGER assistants.

THE following academic changes are announced: Mr. E. I. SHEPARD has returned as instructor in Williams College. Miss C. BUSBEE has been granted leave of absence from Mount Holyoke College, Mass., for further study. Miss E. N. MARTIN has returned to Mount Holyoke as instructor in mathematics. Miss A. L. VAN BENSCHOTEN has returned to her professorship of mathematics at Wells College, Aurora, New York.

DR. F. H. LOUD, professor of mathematics and astronomy at Colorado College since 1877, has retired from active teaching.

PROFESSOR MAURICE LOEWY, director of the Paris observatory and professor of astronomy at the University of Paris, died October 15 at the age of 74 years.

RECENT catalogues of second-hand books: A. Lorentz, 10 Kurprinzstrasse, Leipzig, catalogue 165, logic, about 3,000 titles.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BAIRE (R.). Leçons sur les théories générales de l'analyse. 1er Tome: Principes fondamentaux. Variables réelles. Paris, Gauthier-Villars, 1907. 8vo. 10 + 232 pp. Fr. 8.00

BALL (W. W. R.). Histoire des mathématiques. Edition française revue et augmentée, traduite sur la 3e édition anglaise, par L. Freund. Tome 2: Avec des additions de R. de Montessus, suivi de "Les mathématiques modernes depuis Newton jusqu' à nos jours," note complémentaire de G. Darboux. Paris, Hermann, 1907. 8vo. 278 pp.

CARNOY (J.). Cours de géométrie analytique: géométrie de l'espace. 6e édition, entièrement refondue. Paris, 1907. 8vo. 579 pp. Fr. 9.00

DARBOUX (G.). See BALL (W. W. R.).

FREUND (L.). See BALL (W. W. R.).

KALUZA (T.). Die Tschirnhaustransformation algebraischer Gleichungen mit einer Unbekannten. (Diss.) Königsberg, 1907. 8vo. 103 pp.

LESSER (I.). Die Entwickelung des Funktionsbegriffes und die Pflege des funktionalen Denkens im Mathematikunterricht unserer höheren Schulen. Frankfurt, Knauer, 1907. 74 pp. M. 1.80

LONEY (S. L.). See TANNER (A. S. G.).

MONTESSUS (R. DE). See BALL (W. W. R.).

OSGOOD (W. F.). A first course in the differential and integral calculus. New York, Macmillan, 1907. 12 mo. 8 + 423 pp. Cloth.

SCHULTE-TIGGES (E.). Ellipse und Parabel als Zentralprojektionen des Kreises. Berlin, Reimer, 1907. Cloth. M. 10.00

——. Die Hyperbel als Zentralprojektion des Kreises. Berlin, Reimer, 1907. Cloth. M. 10.00

SCHUMACHER (H.). Ueber eine Riemannsche Funktionsklasse mit zerfallender Thetafunktion. (Diss.) Strassburg, 1907.

SCOTT (C. C.). Cartesian plane geometry. Part I: Analytical conics. London, Dent, 1907. 8vo. 442 pp. Cloth. 5s.

TANNER (A. S. G.) and LONEY (S. L.). Solutions of the examples in Loney's Coördinate geometry. London, Macmillan, 1907. 8vo. 7s. 6d.

II. ELEMENTARY MATHEMATICS.

DASSEN (C. C.). Tratado elemental de algebra. Buenos-Aires, Hermanos, 1907. 8vo. 18 + 528 pp.

DAVISON (C.) and RICHARDS (C. H.). Plane geometry for secondary schools. Cambridge, University Press, 1907. 8vo. 420 pp. Cloth. 4s.

EHRIG (G.). Arithmetik und Algebra. Ein Lehr- und Uebungsbuch der Gleichungslehre für Baugewerkenschulen und verwandte technische und gewerbliche Lehranstalten mit besonderer Berücksichtigung der praktischen Anwendungen. Mit über 1000 Uebungsaufgaben und zahlreichen ausgeführten Rechenbeispielen. Leipzig, Leineweber, 1907. 8vo. 8 + 164 pp. M. 3.60

EXERCICES de géométrie comprenant l'exposé des méthodes géométriques et 2000 questions résolues. 4e édition. Paris, Poussielgue, 1907. 8vo. 21 + 1233 pp.

FOURREY (E.). Curiosités géométriques. Paris, 1907. 8 + 431 pp. Fr. 3.50

GERARD (L.). See NIEWENGLOWSKI (B.).

GORSE (F.). A school algebra course. Part I, to simple simultaneous equations; Part II, factors to quadratic equations; Part III, surds to the binomial theorem. Cambridge, University Press, 1907. 8vo. Part I, 106 pp.; Part II, 102 pp.; Part III, 122 pp. Each 1s.

GUILLEMIN (A.). Tableaux logarithmiques A et B équivalant à des tables de logarithmes à 6 et à 9 décimales, avec notice explicative donnant la théorie et le mode d'emploi des tableaux. Paris, 1907. 8vo. Fr. 4.00

HUPE (A.). See MÜLLER (H.).

KETT (A.). Auflösungen für die Trigonometrie. Für Schule und Praxis herausgegeben. 2te vermehrte und verbesserte Auflage. Neustrelitz, Heydemann, 1907. 8vo. 19 pp. M. 0.60

MITTENZWEY (L.). Mathematische Kurzweil oder 333 Aufgaben, Kunststücke, geistanregende Spiele, verfängliche Schlüsse, Scherze, Ueberraschungen und dergleichen aus der Zahlen- und Formenlehre für jung und alt zur Unterhaltung und Belehrung. 5te Auflage. Leipzig, 1907. 108 pp. M. 1.50

MÜLLER (H.). Die Mathematik auf den Gymnasien und Realschulen. Für den Unterricht dargestellt. Ausgabe B: Für reale Anstalten und Reformschulen, unter Mitwirkung von A. Hupe. 2ter Teil. Die Oberstufe. (Lehraufgabe der Klassen Ober-Sekunda und Prima.) Abteilung I: Planimetrie, Algebra, Trigonometrie und Stereometrie. 3te Auflage. Leipzig, Teubner, 1907. 8vo. 10 + 251 pp. M. 2.80

NIEWENGLOWSKI (B.) et GERARD (L.). Leçons de géométrie élémentaire, conformes aux programmes du 27 juillet 1905, pour les classes de première C et D et de mathématiques A et B. Géométrie plane; géométrie dans l'espace. Paris, Gauthier-Villars, 1907. 8vo. Géométrie plane, 12 + 255 pp.; Géométrie dans l'espace, 334 pp. Fr. 5.00

PETERS (J. M.). Die Summen der ganzen positiven Potenzen der natürlichen Zahlen und ihre Darstellung als Funktionen der Summe dieser Zahlen. Köln, 1907. 4to. 24 pp. M. 1.50

PIETZKER (F.). Lehrgang der Elementar-Mathematik. Teil I: Unterstufe. Leipzig, Teubner, 1907. 8vo. 318 pp. M. 3.20

REIDT (F.). Anleitung zum mathematischen Unterricht an höheren Schulen. 2te Auflage, revidiert und mit Anmerkungen versehen von H. Schotten. Berlin, Grote, 1907. 8vo. 269 pp.

RICHARDS (C. H.). See DAVISON (C.).

SCHOTTEN (H.). See REIDT (F.).

SCHUBERT (H.) und SCHUMPELICK (A.). Arithmetik für Gymnasien. 1^es^ Heft. Für mittlere Klassen. Leipzig, Göschen, 1907. M. 1. 80

——. Ausgewählte Resultate zur Arithmetik für Gymnasien. Leipzig, Göschen, 1907. M. 0. 60

SCHUMPELICK (A.). See SCHUBERT (H.).

TANNER (J. H.). High school algebra. New York, American Book Company, 1907. 8vo. 6 + 346 pp. Cloth. $1.00

TARRY (G.). Tablettes des cotes relatives à la base 20580 des facteurs premiers d'un nombre inférieur à N et non divisible par 2, 3, 5, ou 7. Paris, Gauthier-Villars, 1907. 8vo. Fr. 1.25

YOUNG (E.). A first year's course in geometry and physics. London, Bell, 1907. 8vo. 182 pp. 2s. 6d

III. APPLIED MATHEMATICS.

APPELL (P.) et CHAPPUIS (J.). Leçons de mécanique élémentaire, à l'usage des élèves des classes de mathématiques A et B, conformément aux programmes de 1905. 2^e^ édition, entièrement refondue. 1^re^ partie: Notions géométriques. Cinématique; 2^e^ partie: Dynamique et statique du point. Statique des corps solides. Machines simples. Paris, Gauthier-Villars, 1907. 2 volumes. 16mo. 1^re^ partie, 9 + 192 pp. Fr. 2.75. 2^e^ partie, 244 pp. Fr. 3.25

BARKHAUSEN (H.). Das Problem der Schwingungserregung mit besonderer Berücksichtigung schneller elektrischer Schwingungen. Leipzig, 1907. 4 + 113 pp. M. 4.00

CHAPPUIS (J.). See APPELL (P.).

DESPORTES (E.). Eléments de géométrie descriptive. Nouvelle édition entièrement refondue, conforme aux programmes du 27 juillet 1905; classes de première C et D, et de mathématiques A et B. Paris, Colin, 1907. 8vo. 332 pp. Fr. 4.00

EGGERT (O.). Einführung in die Geodäsie. Leipzig, 1907. 10 + 437 pp. M. 10.00

FRIEDEL (J.). See LAMB (H.).

HAUBER (W.). Statik. Neudrucke. Leipzig, 1907. 1^er^ Teil, 148 pp. 2^er^ Teil, 148 pp. Each, M. 0.80

KONEN (H.). See SCHUSTER (A.).

KUENEN (J. P.). Die Zustandsgleichung der Gase und Flüssigkeiten und die Kontinuitätstheorie. Braunschweig, 1907. 10 + 241 pp. M. 6.50

LAMB (H.). Lehrbuch der Hydrodynamik. Deutsche autorisierte Ausgabe (nach der 3^en^ englischen Auflage) besorgt von J. Friedel. (B. G. Teubner's Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. XXVI Band.) Leipzig, Teubner, 1907. 8vo. 14 + 788 pp. Cloth. M. 20.00

LAVILLE (C.). See VALBREUZE (R. DE).

MARTIN (L. A.). Textbook of mechanics. Vol. II. Kinematics and kinetics. New York, Wiley, 1907. 12mo. 4 + 214 pp. Cloth. $1.50

MÜLLER-BRESLAU (H.). Die graphische Statik der Baukonstruktionen. 2ter Band. 1te Abteilung. 4te, vermehrte Auflage. Stuttgart, 1907. 8 + 484 pp. M. 16.00

PERRY (J.) Applied mechanics. A treatise for the use of students, etc. New edition, revised and enlarged. London, Cassell, 1907. 8vo. 696 pp. Cloth. 7s. 6d.

SCHUSTER (A.). Einführung in die theoretische Optik. Autorisierte deutsche Ausgabe, übersetzt von H. Konen. Leipzig, 1907. 8vo. 14 + 413 pp. Cloth. M. 12.00

VALBREUZE (R. DE) et LAVILLE (C.). Eléments de mécanique et de l'électricité. Paris, Dunod, 1907. 16mo. 6 + 379 pp. Fr. 7.00

WALL (C.). Eléments de mécanique appliquée et d'hydrostatique. Notions sommaires sur la résistance des matériaux et sur la graphique, statique à l'usage des élèves, officiers de la marine et des élèves des écoles de maistrance des arsenaux. Paris, Challamel, 1907. 8vo. 269 pp.